2.1 Killing form

1. Prove that the Killing form of a nilpotent Lie algebra L is identically zero.
2. Prove that a Lie algebra L is solvable iff the radical of its Killing form contains $[L, L]$.
3. Let L be the three dimensional solvable Lie algebra with basis $\{x, y, z\}$:

   $$[x, y] = z, \qquad [x, z] = y, \qquad [y, z] = 0.$$

   Compute the radical of its Killing form.
4. Relative to the standard basis of $\mathfrak{sl}(3, F)$, compute the determinant of κ. Which primes divide it?
5. Let $L = L_1 \oplus \cdots \oplus L_t$ be the decomposition of a semisimple Lie algebra L into its simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the various L_i of the components of x.

2.2 Complete reducibility of representations

1. Let V be an L-module. Prove that V is a direct sum of irreducible submodules if and only if each L-submodule W of V possesses a complement L-submodule W' such that $V = W \oplus W'$.
2. Prove the every irreducible representation of a solvable Lie algebra L over $\mathbf{C}$ is one dimensional.
3. Let V be an L-module. Show that the dual space V^* is an L-module in the way that for $x \in L$, $f \in V^*$, $v \in V$,
$$(x.f)(v) := -f(x.v).$$
4. Let V and W be L-modules. Show that the tensor product space $V \otimes W$ is an L-module in the way that for any $x \in L$, $v \in V$ and $w \in W$,
$$x.(v \otimes w) := x.v \otimes w + v \otimes x.w.$$
5. Let V and W be L-modules.
   (a) Show that the map $\Psi : V^* \otimes W \to \mathrm{Hom}\,(V, W)$ defined on the typical generators
   $$\Psi(\delta \otimes w)(v) := \delta(v)w, \qquad \delta \in V^*,\ w \in W,\ v \in V,$$
   is an isomorphism of vector spaces.
   (b) Show that $\mathrm{Hom}\,(V, W)$ with the following L-action is an L-module:
   $$(x.f)(v) = x.f(v) - f(x.v), \qquad x \in L,\ f \in \mathrm{Hom}\,(V, W),\ v \in V.$$
6. Consider the derivations of a reductive Lie algebra $L = Z(L) \oplus [L, L]$. Let $\delta \in \mathrm{Der}\, L$ be arbitrary.
   (a) Show that $\delta([L, L]) \subset [L, L]$, and there is $x \in [L, L]$ such that $\delta|_{[L,L]} = \mathrm{ad}\, x|_{[L,L]}$.
   (b) Show that $\delta(Z(L)) \subset Z(L)$, and every $\phi \in \mathrm{End}\,(L)$ with $\mathrm{Im}\, \phi \subset Z(L)$ and $\mathrm{Ker}\, \phi \supset [L, L]$ is in $\mathrm{Der}\, L$.
   (c) Show by (a) and (b) that $\mathrm{Der}\, L = \mathrm{End}\,(Z(L)) \oplus \mathrm{ad}\, L$.

2.3 Representation of $\mathfrak{sl}(2, F)$: let $L := \mathfrak{sl}(2, F)$, with F algebraically closed and $\mathrm{char}(F) = 0$.

1. For any $m \in \mathbf{N}$, prove that the $(m+1)$ dimensional L-module $V(m)$ is irreducible. (*Hint: first show that every L-submodule of $V(m)$ must be spanned by some weight spaces.*)
2. Suppose a finite dimensional L-module V is decomposed into a direct sum of irreducible submodules: $V \simeq \sum_{m \in \mathbf{N}} a_m V(m)$, where each $a_m \in \mathbf{N}$. Show that:
   (a) the total number of irreducible summands is
   $$\sum_{m \in \mathbf{N}} a_m = \dim V_0 + \dim V_1;$$
   (b) for $m \in \mathbf{N}$, the number of copies of $V(m)$ in V is
   $$a_m = \dim V_m - \dim V_{m+2}.$$
3. $M = \mathfrak{sl}(3, F)$ contains a copy of L in its upper left-hand 2×2 position. Write M explicitly as direct sum of irreducible L-submodules (M viewed as L-module via the adjoint representation): $V(0) \oplus V(1) \oplus V(1) \oplus V(2)$.
4. Decompose the tensor product of the two L-modules $V(3)$, $V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.
5. The irreducible representation of L of highest weight m can also be realized "naturally", as follows. Let X, Y be a basis for the two dimensional vector space F^2, on which L acts as usual. Let $F[X, Y]$ be the polynomial algebra in two variables, and extend the action of L to $F[X, Y]$ by the derivation rule:$z.fg = (z.f)g + f(z.g)$, for $z \in L$, $f, g \in F[X, Y]$. Show that this extension is well defined and that $F[X, Y]$ becomes an L-module. Then show that the subspace of homogeneous polynomials of degree m, with basis X^m, $X^{m-1}Y$, $\cdots$, XY^{m-1} , Y^m, is invariant under L and irreducible of highest weight m.

2.4 Root space decomposition

1. Suppose V is a finite dimensional vector space over $\mathbf{C}$. Let $\mathcal{S} \subseteq \mathrm{End}\,(V)$ be a commuting family of semisimple endomorphisms. Prove that the endomorphisms in $\mathcal{S}$ are simultaneously diagonalizable. *[Hint: Let $\{h_1, h_2, \cdots, h_m\}$ be a maximal linearly independent subset of $\mathcal{S}$. Then V is a direct sum of eigenspaces V_λ of h_m associate to eigenvalues λ. Show that every $h \in \mathcal{S}$ acts invariantly on each V_λ. Then proceed the induction on m.]*
2. For each classical linear Lie algebra of types A_ℓ, B_ℓ, C_ℓ, D_ℓ, determine the roots and root spaces.
3. Show that all Cartan integers of $\mathfrak{sl}(\ell+1, \mathrm{F})$ are $0, \pm 1$.
4. Compute the basis of $\mathfrak{sl}(\ell+1, \mathrm{F})$ which is dual (via the Killing form κ) to the standard basis $\{h_i \mid i \in [\ell]\} \cup \{e_{ij} \mid i, j \in [\ell+1],\ i \neq j\}$ of $\mathfrak{sl}(\ell+1, \mathrm{F})$. *[Hint: Recall that the root spaces $L_\alpha \perp L_\beta$ whenever $\alpha + \beta \neq 0$, and $\kappa(x, y) = 2(\ell+1)\mathrm{Tr}(xy)$ for $x, y \in \mathfrak{sl}(\ell+1, \mathrm{F})$.]*
5. Let H be a maximal toral subalgebra of a semisimple Lie algebra L. For $\alpha \in H^*$, denote $t_\alpha \in H$ such that $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$. Prove that $t_{a\alpha + b\beta} = a t_\alpha + b t_\beta$ for any $\alpha, \beta \in H^*$ and $a, b \in \mathrm{F}$.