3.1 Introduction of Root Systems

1. Prove that $\mathcal{W}$ is a normal subgroup of Aut (Φ).
2. Prove that the respective Weyl groups of $A_1 \times A_1$, A_2 , B_2, G_2 are dihedral of order 4, 6, 8, 12.
3. Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) < 0$.
4. Let $\alpha, \beta \in \Phi$ span a two dimensional subspace E' of E. Prove that $E' \cap \Phi$ is a root system in E', so that it must be one of the four root systems in $\mathbf{R}^2$ (see Figure 1 of Section 9.4 in the textbook).
5. Let c be a positive real number. If Φ possesses any roots of squared length c, prove that the set of all such roots is a root system in the subspace of E it spans.

3.2 Simple Roots and Weyl Group

1. For each of the rank 2 root systems, sketch all the roots, label a base Φ, and sketch the corresponding fundamental Weyl chamber.
2. When Φ is irreducible, prove that each closed Weyl chamber contains exactly one root of each root length.
3. Let $\lambda \in \mathcal{C}(\Delta)$, if $\sigma\lambda = \lambda$ for some $\sigma \in \mathcal{W}$, then $\sigma = 1$.
4. Prove that there is a unique element σ in $\mathcal{W}$ sending Φ^+ to Φ^- (relative to Δ). Prove that any reduced expression for σ must involve all σ_α $(\alpha \in \Delta)$. Discuss $\ell(\sigma)$.