Hodrick-Prescott Filter

Hyeongwoo Kim

March 12, 2004

Let's suppose that the original series y_t is composed of a trend component (τ_t) and a cyclical component (c_t). That is,

$$y_t = \tau_t + c_t \ , \ t = 1, 2, \cdots, T$$

Hodrick and Prescott (1997) suggest a way to isolate c_t from y_t by following minimization problem.

$$Min_{\{\tau_t\}_{t=1}^{T}} \left[\sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} (\boldsymbol{\nabla}^2 \tau_{t+1})^2 \right] \tag{1}$$

, where λ is the penalty parameter. The first term in the loss function, (1) penalizes the variance of c_t, while the second term puts a prescribed penalty to the lack of smoothness in τ_t. Put it differently, the HP filter identifies the cyclical component c_t from y_t by the trade-off to the extent to which the trend component keeps track of the original series y_t (good fit) against the prescribed smoothness in τ_t. Note that as λ approaches to 0, the trend component becomes equivalent to the original series, while as λ diverge to ∞, τ_t approaches to the linear trend.

It is customary to set λ to 1,600 for quarterly data. For monthly and annual data, Maravall and del Rio (2001) recommended to use $100,000 < \lambda_M < 140,000$ and $6 < \lambda_A < 14$, respectively.

By taking derivatives of the loss function (1) with respect to τ_t , $t = 1, \cdots, T$ and rearranging them, it can be shown that the solution to (1) can be written as following matrix form.

$$\mathbf{y}_T = (\lambda \mathbf{F} + \mathbf{I}_T) \boldsymbol{\tau}_T \tag{2}$$

, where $\mathbf{y}_T$ is the $(T \times 1)$ vector of original series and

$$\mathbf{F} = \begin{bmatrix}
1 & -2 & 1 & 0 & \cdots & & & & & & & & \cdots & 0 \\
-2 & 5 & -4 & 1 & 0 & \cdots & & & & & & & \cdots & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & & & & \cdots & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & & & & \vdots \\
\vdots & \ddots & & & & & & \ddots & & & & & & \\
& & & & & & & \ddots & & & & & \ddots & \vdots \\
0 & & & & & & & & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\
\vdots & & & & & & & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\
& & & & & & & & & \cdots & 0 & 1 & -4 & 5 & -2 \\
0 & \cdots & & & & & & & & & \cdots & 0 & 1 & -2 & 1
\end{bmatrix}$$

Then, the trend component and the cyclical component can be identified as follows.

$$\boldsymbol{\tau}_T = (\lambda \mathbf{F} + \mathbf{I}_T)^{-1} \mathbf{y}_T \tag{3}$$

$$\mathbf{c}_T = \mathbf{y}_T - \boldsymbol{\tau}_T \tag{4}$$

Reference:

1. Hodrick, Robert, and Edward C. Prescott (1997), "Postwar U.S. Business Cycles: An Empirical Investigation," Journal of Money, Credit, and Banking.

2. Maravall, Agustín, and Ana del Rio (2001), "Time Aggregation and the Hodrick-Prescott Filter," Banco de España.

3. Mark, Nelson C. (2001), *International Macroeconomics and Finance*.