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STABLE DETERMINATION OF A BOUNDARY COEFFICIENT IN AN ELLIPTIC EQUATION

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We prove a logarithmic stability estimate for a Cauchy problem associated with a second order elliptic operator. Our proof is essentially based on a Carleman estimate by A. L. Bukhgeim. This result is applied to establish a stability estimate for the inverse problem of determining a boundary coefficient (or a boundary function) by a single boundary measurement. This kind of inverse problems is motivated by the corrosion detection problem.

Keywords: Carleman estimate; logarithmic stability estimate; corrosion detection.

AMS Subject Classification: 35R30, 35J05

1. Introduction

One of the models describing the electrostatics of a conductor Ω having an inaccessible part of his boundary, denoted by Γ, affected by corrosion is given by the following boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_\nu u = f, & \text{on } \gamma, \\ \partial_\nu u + qu = 0, & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where ν is the unit outer normal to $\partial\Omega$ and $\gamma \cup \Gamma = \partial\Omega$.

In (1.1), u represents the electrostatic potential, f is the prescribed current density on the accessible part of the boundary γ; while q, called the corrosion coefficient, represents the characteristic of corrosion damage. Of course, there are also several other models, including nonlinear ones (see for instance Ref. 13).

In the sequel, we assume that γ and Γ are two closed subsets of $\partial\Omega$ with nonempty interior.

The corrosion detection problem consists of the determination of the coefficient $q(x)$ by measuring the corresponding boundary voltage $g = u_{|\gamma}$ on the accessible part of $\partial\Omega$. By the uniqueness of the Cauchy problem for the Laplace equation, we can easily prove that there is at most one solution for this inverse problem. The purpose of our paper is the stability issue for the above-mentioned inverse problem. It should be remarked here that this kind of stability estimate implies the convergence rate of the Tikhonov regularized solutions.[6]

In the following, we assume that $(\int_\gamma [u^2 + |\nabla u|^2]d\sigma)$ is small enough. Since we will use the following stability results to estimate the difference between the exact solution and the numerical results, this assumption is reasonable.

We fix f and for $i = 1, 2$, let q_i be given and $g_i = u_{i|\gamma}$, where u_i is the solution of the boundary value problem (1.1) when $q = q_i$. Let us assume for the moment that Ω, considered as an open bounded subset of $\mathbb{R}^2$, and f are chosen in such a way that u_i exists and it is smooth enough. Since

$$\partial_\nu u_1 + q_1 u_1 = \partial_\nu u_2 + q_2 u_2 \quad \text{on } \Gamma,$$

we have

$$(q_1 - q_2)u_1 = q_2(u_2 - u_1) + \partial_\nu(u_2 - u_1) \quad \text{on } \Gamma. \tag{1.2}$$

Let K be a compact subset of $\{x \in \Gamma;\ u_1 \neq 0\}$ and assume that $0 \leq q_2 \leq M$ on Γ, where M is some positive constant. Then (1.2) implies

$$\|q_1 - q_2\|_{L^2(K)} \leq C(\|u_2 - u_1\|_{L^2(\Gamma)} + \|\partial_\nu(u_2 - u_1)\|_{L^2(\Gamma)}),$$

for some positive constant C.

Therefore it is easy to see that, if we are able to prove the following form of estimate for the Cauchy problem for the Laplace equation

$$\left(\int_\Gamma [u^2 + |\nabla u|^2]d\sigma\right)^{\frac{1}{2}} \leq A\left\{\ln\left(\frac{B}{(\int_\gamma [u^2 + |\nabla u|^2]d\sigma)^{\frac{1}{2}}}\right)\right\}^{-1}, \tag{1.3}$$

where $u = u_1 - u_2$, and A, B are two positive constants which are independent of u, then we can get

$$\|q_1 - q_2\|_{L^2(K)} \leq A\left\{\ln\left(\frac{B}{(\int_\gamma [u^2 + |\nabla u|^2]d\sigma)^{\frac{1}{2}}}\right)\right\}^{-1}.$$

If we assume that $\partial_\nu u = 0$ on γ, it can be obtained that

$$\|q_1 - q_2\|_{L^2(K)} \leq A \left\{ \ln \left(\frac{B}{(\int_\gamma [(g^1 - g^2)^2 + |\partial_\tau (g^1 - g^2)|^2] d\sigma)^{\frac{1}{2}}} \right) \right\}^{-1}, \tag{1.4}$$

where ∂_τ is the derivative along the unit tangent vector on $\partial\Omega$.

This paper is organized as follows: In Sec. 2, we state our main results and give some remarks. In Sec. 3, the proof of the stability estimate for the Cauchy problem for the second-order elliptic equation is presented. In Sec. 4, we prove a stability estimate when the domain is a square. In the last section, we present a numerical example for the inverse problem of determining the coefficient q.

2. Main Results

In the sequel, Ω denotes a bounded open subset of $\mathbb{R}^2$ of class $C^{2,\alpha}$ for some α, $0 < \alpha < 1$. We consider the following second-order differential operator:

$$P = -\Delta + b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + c,$$

where b_1, b_2 and c belong to $L^\infty(\Omega)$.

We are now ready to state our main result.

Theorem 2.1. *Let $M > 0$ be a given constant. If $(\int_\gamma [w^2 + |\nabla w|^2] d\sigma)$ is sufficiently small, then there exist three positive constants ϵ, A and B (depending on Γ, M and L^∞-norms of the coefficients of P) such that*

$$\left(\int_\Gamma [w^2 + |\nabla w|^2] d\sigma \right)^{\frac{1}{2}} \leq A \left\{ \ln \left(\frac{B}{(\int_\gamma [w^2 + |\nabla w|^2] d\sigma)^{\frac{1}{2}}} \right) \right\}^{-1}$$

for all $w \in C^2(\overline{\Omega})$ satisfying

$$Pw = 0, \quad \|w\|_{C^2(\overline{\Omega})} \leq M.$$

This theorem will be proved in the next section.

Next we give the exact statement of the logarithmic stability estimate for the corrosion detection problem. To this end, we rewrite the boundary value problem (1.1) in the following form:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ \partial_\nu u + qu = f, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We make the following assumptions:

(a) $q \in C^{1,\alpha}(\partial\Omega)$, $q \geq 0$ and non identically equal to zero,
(b) $f \in C^{1,\alpha}(\partial\Omega)$.

Under these assumptions, by the theory of partial differential equations, the boundary value problem (2.1) has a unique solution $u = u_q \in C^{2,\alpha}(\overline{\Omega})$ (see Theorem 6.31 in Ref. 11 and the remark following it). Moreover, according to the classical Hölder *a priori* estimate (see again Theorem 6.30 of Ref. 11), one can easily prove that the mapping $q \in C^{1,\alpha}(\partial\Omega) \to u_q \in C^{2,\alpha}(\overline{\Omega})$ is continuous (more precisely one can show that this mapping is locally Lipschitz continuous). From the previous discussion and Theorem 2.1, we have

Corollary 2.2. *Let $M > 0$ be a given constant. For $i = 1, 2$, let q_i satisfy* (*a*), $\operatorname{supp}(q_i) \subset \Gamma$ *and* $\|q_i\|_{C^{1,\alpha}(\partial\Omega)} \leq M$. *Assume that f satisfies* (*b*) *and is non identically equal to zero. Let K be a compact subset of $\{x \in \Gamma;\ u_1 \neq 0\}$ and Γ_0 be an open subset of $\partial\Omega\backslash\Gamma$. If $\|q_1 - q_2\|_{C^{1,\alpha}(\partial\Omega)}$ is sufficiently small, then there exist positive constants A, B* (*depending on* Γ, Γ_0 *and* M) *such that*

$$\|q_1 - q_2\|_{L^2(K)} \leq A\left\{\ln\left(\frac{B}{\|g_1 - g_2\|_{L^2(\Gamma_0)}}\right)\right\}^{-1},$$

where $g_i = u_{q_i|\Gamma_0}$.

Note that we first obtain estimate (1.4) from Theorem 2.1. Next, $\|\partial_\tau(g_1 - g_2)\|_{L^2(\gamma)}$ is estimated in terms of $\|g_1 - g_2\|_{L^2(\gamma)}$ as follows: by the classical Hölder *a priori* estimate (e.g. Ref. 11) we get

$$\|u_1 - u_2\|_{C^2(\overline{\Omega})} \leq C_0,$$

where C_0 is some constant depending on Ω, M and f. Next, it follows from the well-known interpolation inequalities for the Sobolev spaces

$$\begin{aligned}\|\partial_\tau(g_1 - g_2)\|_{L^2(\gamma)} &\leq \|g_1 - g_2\|_{H^1(\gamma)} \leq C_1\|g_1 - g_2\|_{L^2(\gamma)}^{\frac{1}{2}}\|g_1 - g_2\|_{H^2(\gamma)}^{\frac{1}{2}} \\ &\leq C_2\|g_1 - g_2\|_{L^2(\gamma)}^{\frac{1}{2}}\|g_1 - g_2\|_{C^2(\overline{\Omega})}^{\frac{1}{2}},\end{aligned}$$

and then

$$\|\partial_\tau(g_1 - g_2)\|_{L^2(\gamma)} \leq C\|g_1 - g_2\|_{L^2(\gamma)}^{\frac{1}{2}},$$

for some positive constant C depending on Ω, M and f.

Remark 1. Concerning other kinds of corrosion detection problems, various logarithmic stability estimates by different methods were already established in Refs. 1, 3 and 7.

Remark 2. For the solution u_1 of problem (1.1), $\{x \in \Gamma;\ u_1(x) \neq 0\}$ is an open dense subset of Γ. Indeed, if this is not true then we find Γ' an open subset of Γ such that $u_1 = 0$ on Γ' and then $\partial_\nu u_1 = 0$ on Γ'. Hence u_2 is identically equal to zero, according to the unique continuation property, and so is f. But this contradicts the fact that f is assumed to be non-identically equal to zero.

Remark 3. From Proposition 2.1 in Ref. 8, we know that, under the assumptions of Corollary 1.1, if f is non-negative and non-identically equal to zero then $u_{q_1} > 0$

on $\partial\Omega$. In this case we can replace the term in the estimate above $\|q_1 - q_2\|_{L^2(K)}$ by $\|q_1 - q_2\|_{L^2(\Gamma)}$. In general, without the non-negativity of f we cannot hope to get more than $\|q_1 - q_2\|_{L^2(K)}$ because we do not know how to characterize the set of zeros of u_q on Γ. However, under an appropriate condition on f, Alessandrini *et al.*[1] prove that u_q has at most a finite number of zeros on any subset Γ_d of Γ, where

$$\Gamma_d = \partial(\Omega_\Gamma)_d \cap \Gamma, \quad (\Omega_\Gamma)_d = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega \setminus \Gamma) > d\}$$

(see Proposition 2.3 in Ref. 1 for the exact statement).

Remark 4. In the inverse problem of detecting corrosion on the interior boundary of a pipe, the domain Ω is a spherical shell (see Ref. 5). In this case, by a simple method based on a Fourier analysis, we can prove the conclusion of Corollary 2.2 (see Ref. 9 for more details).

As we mentioned above, there are some models in corrosion detection which are nonlinear (see Ref. 13 for more details). By our stability estimate for the Cauchy problem for the second-order elliptic operators, we can get the similar estimations.

One of these models is the following boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ \partial_\nu u = f, & \text{on } \gamma \\ \partial_\nu u + g(u) = 0, & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Let us assume that γ and Γ are disjoint. Pick $0 \le \varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi = 1$ in Γ and $\varphi = 0$ on γ. Then the boundary value problem (2.2) can be rewritten in the following form:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ \partial_\nu u + \varphi(x)g(u) = (1 - \varphi(x))f, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

From Theorems 17.28 and 17.30 in Ref. 11 we deduce that under the following conditions:

(a$'$) $g \in \mathcal{G} = \{h \in C^{3,\alpha}(\mathbb{R});\ h' \ge 0\}$,
(b$'$) $f \in C^{3,\alpha}(\gamma)$ and is non-identically equal to zero,

the boundary value problem (2.3) has a unique solution $u_g \in C^{2,\alpha}(\overline{\Omega})$. In addition, the classical Hölder *a priori* estimate (for linear boundary value problems, see Theorem 6.30 in Ref. 11) shows that $g \in \mathcal{G} \to u_g \in C^{2,\alpha}(\overline{\Omega})$ is continuous when $\mathcal{G}$ is equipped with the topology of $C^{1,\alpha}$.

Remark 5. Since we use a general existence theorem we expect that assumptions (a') and (b') can be relaxed.

Now let $g_i \in \mathcal{G}$, $i = 1, 2$ and denote by u_i the solution of the boundary value problem (2.2) when $g = g_i$. Then a straightforward computation shows that

$$g_1(u_1) - g_2(u_1) = \partial_\nu(u_2 - u_1) + (g_2(u_2) - g_2(u_1)) \quad \text{on } \Gamma. \tag{2.4}$$

We denote $a = \min_\Gamma u_1$ and $b = \max_\Gamma u_1$ and let $t \in [a, b]$ such that $|g_2(t) - g_1(t)| = \max_{[a,b]} |g_2 - g_1|$ and let $x_0 \in \Gamma$ satisfy $u_1(x_0) = t$. By a continuity argument we find that there exists a ball B centered at x_0 such that $|g_2(u_1) - g_1(u_1)| \geq \frac{1}{2} \max_{[a,b]} |g_2 - g_1|$ on $\Gamma_0 = \Gamma \cap B$. Consequently

$$\|g_1 - g_2\|_{C([a,b])} \leq \frac{2}{|\Gamma_0|^{\frac{1}{2}}} \|g_1(u_1) - g_2(u_1)\|_{L^2(\Gamma)}.$$

This and (2.4) imply

$$\|g_1 - g_2\|_{C([a,b])} \leq \frac{2}{|\Gamma_0|^{\frac{1}{2}}} \max\{1, M\}(\|\partial_\nu(u_1 - u_2)\|_{L^2(\Gamma)} + \|u_1 - u_2\|_{L^2(\Gamma)}) \quad (2.5)$$

provided that $\|g_2'\|_{L^\infty(\mathbb{R})} \leq M$ for some positive constant M.

We introduce then $\mathcal{G}_M$, the subset of functions $g \in \mathcal{G}$ such that $\|g'\|_{L^\infty(\mathbb{R})} \leq M$. Since $u_1 - u_2$ satisfies

$$\begin{cases} \Delta(u_1 - u_2) = 0 & \text{in } \Omega, \\ \partial_\nu(u_1 - u_2) = 0 & \text{on } \gamma, \end{cases}$$

in view of (2.5), we get as an immediate consequence of Theorem 2.1:

Corollary 2.3. *We assume that γ and Γ are disjoint. For $i = 1, 2$, let $g_i \in \mathcal{G}_M$ and assume that f satisfies (b'). Let $a = \min_\Gamma u_1$ and $b = \max_\Gamma u_1$. If $\|g_1 - g_2\|_{C^{1,\alpha}(\mathbb{R})}$ is sufficiently small, then there exist positive constants A, B (depending on Γ, M, g_1 and $g_1 - g_2$) such that*

$$\|g_1 - g_2\|_{C([a,b])} \leq A \left\{ \ln \left(\frac{B}{\|h_1 - h_2\|_{L^2(\gamma)}} \right) \right\}^{-1},$$

where $h_i = u_{g_i}{}_{|\gamma}$.

3. Proof of Theorem 2.1

We need the following two lemmas.

Lemma 3.1.[2] *Let ψ be an arbitrary function in $C^2(\overline{\Omega})$. Then the following Carleman estimate holds*

$$\int_\Omega (\Delta\psi u^2 + (\Delta\psi - 1)|\nabla u|^2) e^\psi dx \leq \int_\Omega |\Delta u|^2 e^\psi dx + \int_{\partial\Omega} [\partial_\nu \psi (u^2 + |\nabla u|^2) + 2|\partial_\tau |\nabla u|^2|] e^\psi d\sigma,$$

for all $u \in C^2(\overline{\Omega})$.

Lemma 3.2. *Let Γ and γ be as in Theorem 2.1. Then there exists $\psi_0 \in C^2(\overline{\Omega})$ with the following properties:*

$$\Delta\psi_0 = 0, \quad \text{in } \Omega; \qquad \psi_0 = 0, \quad \text{on } \Gamma;$$
$$\partial_\nu \psi_0 < 0, \quad \text{on } \Gamma; \qquad \psi_0 \geq 0, \quad \text{on } \gamma.$$

Proof. Let $\chi \in C^{2,\alpha}(\partial\Omega)$ such that

$$\chi = 0, \quad \text{on } \Gamma; \qquad \chi \geq 0 \quad \text{on } \gamma$$

and χ is non-identically equal to zero on γ.

Since Ω is of class $C^{2,\alpha}$, the following boundary value problem:

$$\begin{cases} \Delta\psi_0 = 0, & \text{in } \Omega, \\ \psi_0 = \chi, & \text{on } \partial\Omega, \end{cases}$$

has the unique solution $\psi_0 \in C^{2,\alpha}(\overline{\Omega})$ (see for instance Ref. 11). ψ_0 is not a constant because χ is non-identically equal to zero. Hence $\psi_0 > 0$ in Ω according to the strong maximum principle. But ψ_0 is equal to zero at each point of Γ. We can then apply Hopf's lemma to deduce that $\partial_\nu\psi_0 < 0$ on Γ. ψ_0 satisfies then the required properties. □

Proof of Theorem 2.1. The proof is based on the Carleman estimate in Lemma 3.1 with an appropriate choice of the weight function ψ. Let $\psi_0 \in C^2(\overline{\Omega})$ be non-identically equal to zero and satisfy

$$\Delta\psi_0 = 0, \quad \text{in } \Omega; \qquad \psi_0 = 0, \quad \text{on } \Gamma;$$
$$\partial_\nu\psi_0 < 0, \quad \text{on } \Gamma; \qquad \psi_0 \geq 0 \quad \text{on } \gamma.$$

By Lemma 3.2, such a function exists. Let λ be a positive real number to be chosen later. We denote by $\psi_1 \in C^2(\overline{\Omega})$ the unique solution (see Ref. 11) of the following boundary value problem:

$$\begin{cases} \Delta\psi_1 = \lambda, & \text{in } \Omega, \\ \psi_1 = 0, & \text{on } \partial\Omega. \end{cases}$$

Let $s \geq 1$, $w \in C^2(\overline{\Omega})$ such that $Pw = 0$. Then an application of the Carleman estimate in Lemma 3.1 with $\psi = \psi_1 + s\psi_0$ gives

$$\begin{aligned} &\int_\Omega (\lambda w^2 + (\lambda - 1)|\nabla w|^2)e^\psi dx \\ &\quad \leq \int_\Omega |\Delta w|^2 e^\psi dx + \int_{\partial\Omega} [\partial_\nu\psi(w^2 + |\nabla w|^2) + 2|\partial_\tau|\nabla w|^2|]e^\psi d\sigma. \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned} \int_\Omega |\Delta w|^2 e^\psi dx &\leq 4\int_\Omega (Pw)^2 e^\psi dx + 4\|c\|^2_{L^\infty(\Omega)} \int_\Omega w^2 e^\psi dx \\ &\quad + 4\max(\|b_1\|^2_{L^\infty(\Omega)}, \|b_2\|^2_{L^\infty(\Omega)}) \int_\Omega |\nabla w|^2 e^\psi dx \\ &\leq 4\max(\|b_1\|^2_{L^\infty(\Omega)}, \|b_2\|^2_{L^\infty(\Omega)}) \int_\Omega |\nabla w|^2 e^\psi dx \\ &\quad + 4\|c\|^2_{L^\infty(\Omega)} \int_\Omega w^2 e^\psi dx. \end{aligned} \tag{3.2}$$

We fix λ such that

$$\lambda \geq 4\max(\|b_1\|^2_{L^\infty(\Omega)}, \|b_2\|^2_{L^\infty(\Omega)}) + 1$$

and

$$\lambda \geq 4\|c\|^2_{L^\infty(\Omega)}.$$

Then (3.1) and (3.2) imply

$$0 \leq \int_{\partial\Omega} [\partial_\nu \psi(w^2 + |\nabla w|^2) + 2|\partial_\tau |\nabla w|^2|] e^\psi d\sigma. \tag{3.3}$$

We assume in addition that

$$\|w\|_{C^2(\overline{\Omega})} \leq M,$$

where $M > 0$ is a given constant. By using $\psi_0 = 0$ on Γ and $\theta = \min_\Gamma |\partial_\nu \psi_0| > 0$, we deduce from (3.3)

$$0 \leq -\frac{s\theta}{2}\int_\Gamma (w^2 + |\nabla w|^2) d\sigma + 4M^2 + sK,$$

where

$$K = C_0 \int_\gamma (w^2 + |\nabla w|^2) e^\psi d\sigma + 2M \int_\gamma |\nabla w| e^\psi d\sigma.$$

Here C_0 is a constant depending on Γ. That is

$$\int_\Gamma (w^2 + |\nabla w|^2) d\sigma \leq C_1 \left(\frac{1}{s} + K\right), \tag{3.4}$$

for some positive constant C_1 depending on M and Γ.

We set $\delta = \int_\gamma (u^2 + |\nabla u|^2) d\sigma$. Then a straightforward computation shows

$$K \leq C_2 e^{ks}(\delta + \sqrt{\delta}) \leq C_2 e^{ks}\sqrt{\delta} \quad \text{if } \delta \leq 1.$$

Here C_2 is a constant depending on Γ, M and L^∞-norms of the coefficients of P and k depends on Γ. The last estimate combined with (3.4) gives

$$\int_\Gamma (w^2 + |\nabla w|^2) d\sigma \leq C_3 \min_{s \geq 1} \left(\frac{1}{s} + e^{ks}\sqrt{\delta}\right), \tag{3.5}$$

where C_3 is a constant depending on Γ, M and L^∞-norms of the coefficients of P.

An elementary calculation shows that the minimum is attained at s_* such that

$$\sqrt{\delta} = \frac{e^{-ks_*}}{ks_*^2}.$$

We note that, since $s \to \frac{e^{-ks}}{s^2}$ is non-increasing, $s_* \geq 1$ if δ is small enough. In view of (3.5), we find

$$\int_\Gamma (w^2 + |\nabla w|^2) d\sigma \leq C_1 \left(\frac{1}{s_*} + \frac{1}{ks_*^2}\right) \leq C_1 \left(1 + \frac{1}{k}\right)\frac{1}{s_*}. \tag{3.6}$$

Or

$$\frac{1}{\sqrt{\delta}} = ks_*^2 e^{ks_*} \le 2ke^{(k+1)ks_*}.$$

That is

$$\frac{1}{s_*} \le \frac{k+1}{\ln\left(\frac{1}{2k\sqrt{\delta}}\right)}. \tag{3.7}$$

The desired estimate follows then from a combination of (3.6) and (3.7). The proof is complete. □

4. Some Further Results

The method mentioned in Sec. 2, based on a Carleman estimate, does not work if the domain Ω is not sufficiently smooth. In this section, we use a method, based on a Fourier analysis, to establish a stability estimate when the domain Ω is a square.

First we consider the Cauchy problem for the Laplace equation in a square domain. Let $\Omega = (0,1) \times (0,1)$, $\Gamma = [0,1] \times \{0\}$, $\gamma = \partial\Omega \backslash \Gamma$.

The Cauchy problem for the Laplace equations in Ω is formulated as

$$\begin{cases} \Delta w = 0, & \text{in } \Omega \\ w, \partial_\nu w & \text{are given on } \gamma. \end{cases}$$

We set

$$v_\pm(\xi)(x,y) = e^{-ix\xi \pm \xi y}, \quad \xi \in \mathbb{R}$$

and

$$f(\xi) = \int_0^1 w(x,0)e^{-ix\xi}dx, \quad g(\xi) = \int_0^1 \partial_y w(x,0)e^{-ix\xi}dx.$$

If we apply Green's formula to $w \in C^2(\overline{\Omega})$, with $\Delta w = 0$ in Ω, and $v_\pm(\xi)$, then we find

$$-g(\xi) + \xi f(\xi) = \int_\gamma w\partial_\nu v_+(\xi)d\sigma - \int_\gamma \partial_\nu w v_+(\xi)d\sigma,$$

$$-g(\xi) - \xi f(\xi) = \int_\gamma w\partial_\nu v_-(\xi)d\sigma - \int_\gamma \partial_\nu w v_-(\xi)d\sigma.$$

The last two identities imply

$$g(\xi) = \frac{1}{2}\left[\int_\gamma (-w\partial_\nu v_+(\xi) - \partial_\nu w v_+(\xi) + w\partial_\nu v_-(\xi) - \partial_\nu w v_-(\xi))d\sigma\right].$$

Since $|v_\pm(\xi)| \le e^{|\xi|}$ and $|\partial_\nu v_\pm(\xi)| \le |\xi|e^{|\xi|}$, we easily show the estimate

$$|g(\xi)| \le e^{2|\xi|}\left(\int_\gamma |\partial_\nu w|d\sigma + \int_\gamma |w|d\sigma\right). \tag{4.1}$$

Note that g is the Fourier transform of $\partial_y w(\cdot,0)$ extended by 0 outside $(0,1)$, we derive (see for instance Ref. 7) from (4.1) that

$$\|\partial_y w(\cdot,0)\|_{L^2(0,1)} \le A\left\{\ln \frac{B}{\int_\gamma |\partial_\nu w|d\sigma + \int_\gamma |w|d\sigma + |w(0,0)| + |w(1,0)|}\right\}^{-1} \tag{4.2}$$

provided that $\|\partial_y w(\cdot,0)\|_{H^1(0,1)} \le M$ and $(\int_\gamma |\partial_\nu w|d\sigma + \int_\gamma |w|d\sigma + |w(0,0)| + |w(1,0)|)$ is small enough. Here A and B are some positive constants depending on M.

In addition, if we assume that $\partial_\nu w = 0$ on Γ and $\|w(\cdot,0)\|_{H^1(0,1)} \le M'$, then by estimate (3.9) in Ref. 7 there exist two positive constants A' and B' depending on M' such that

$$\|w(\cdot,0)\|_{L^2(0,1)} \le A'\left\{\ln \frac{B'}{\int_\gamma |w|d\sigma + |w(0,0)| + |w(1,0)|}\right\}^{-1}, \tag{4.3}$$

when $\int_\gamma |w|d\sigma + |w(0,0)| + |w(1,0)|$ is small enough.

Let us consider the following boundary value problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ -\partial_y u(x,0) + q(x)u(x,0) = 0, & x \in (0,1), \\ \partial_y u(x,1) = f(x), & x \in (0,1), \\ \partial_x u(0,y) = \partial_x u(1,y) = 0, & y \in (0,1), \end{cases} \tag{4.4}$$

and we recall (see Ref. 12) that given a q in

$$\mathcal{Q} = \{p \in C^3[0,1] | p \ge 0, p \ne 0, \mathrm{supp}(p) \subset (0,1)\}$$

and let f satisfy

$$f \in C^3[0,1], \quad \mathrm{supp}(f) \subset (0,1),$$

the boundary value problem (4.4) has a unique solution $u \in C^2(\overline{\Omega})$.

Let u_i be the solution of (4.4) corresponding to $q = q_i$, $i = 1, 2$. The following result improves Theorem 3.2 of Ref. 7.

Proposition 4.1. *Assume that* q_m, $q_M \in \mathcal{Q}$, $f \in C^3[0,1]$ *with* $\mathrm{supp}(f) \subset (0,1)$ *and* $M > 0$ *is given. We assume that* f *is non-negative and non-identically equal to zero. Then there exist positive constants* A, B *such that, for each* q_1, $q_2 \in \mathcal{Q}$ *which satisfy*

- $\|q_1\|_{W^{2,\infty}(0,1)}$, $\|q_1\|_{W^{2,\infty}(0,1)} \le M$
- $q_m \le q_1,\ q_2 \le q_M$,

it holds that

$$\|q_1 - q_2\|_{L^2(0,1)} \leq A\left\{\ln\left(\frac{B}{\int_\gamma |u_1 - u_2| d\sigma + |(u_1 - u_2)(0,0)| + |(u_1 - u_2)(1,0)|}\right)\right\}^{-1}$$

provided that $\|q_1 - q_2\|_{W^{1,\infty}(0,1)}$ *is sufficiently small. Here the constants* A, B *depend on* q_m, q_M, M *and* f.

Proof. (sketch) Let $u = u_1 - u_2$, we will prove below the following estimate:

$$\|u_i(\cdot, 0)\|_{H^2(0,1)}, \quad \|\partial_y u_i(\cdot, 0)\|_{H^1(0,1)} \leq C, \quad i = 1,\ 2, \tag{4.5}$$

for some positive constant C depending on q_m, M and f.

In view of these estimates, we deduce from (4.2) and (4.3) that there exist positive constants A, B such that

$$\|u(\cdot, 0)\|_{L^2(0,1)} + \|\partial_y u(\cdot, 0)\|_{L^2(0,1)} \leq A\left\{\ln \frac{B}{\int_\gamma |u| + |u(0,0)| + |u(1,0)|}\right\}^{-1}$$

provided $\int_\gamma |u| + |u(0,0)| + |u(1,0)|$ is small enough.

The rest of the proof is similar to that of Theorem 3.2 in Ref. 7.

The proof is complete. □

Proof of (4.5). For simplicity we set $U = u_i$, and $Q = q_i$, $i = 1$ or 2. From Proposition 3.2 of Ref. 7 we have

$$\|U(\cdot, 0)\|_{H^1(0,1)} \leq C_0, \tag{4.6}$$

where C_0 is some positive constant depending on q_m, M and f.

In order to prove the H^2 estimate for $U(\cdot, 0)$ we introduce $z = \partial_x^2 U$. It is straightforward to check that z is the solution of the following boundary value problem

$$\begin{cases} \Delta z = 0, & \text{in } \Omega, \\ -\partial_y z(x,0) + Q(x) z(x,0) = -2Q'(x)\partial_x U(\cdot,0) - Q''(x) U(\cdot, 0), & x \in (0,1), \\ \partial_y z(x,1) = f''(x), & x \in (0,1), \\ z(0,y) = z(1,y) = 0, & y \in (0,1). \end{cases}$$

In a classical way, we obtain from an application of Green's formula

$$\int_\Omega |\nabla z|^2 + \int_0^1 Qz^2(\cdot,0) = -2\int_0^1 Q'\partial_x U(\cdot,0)z(\cdot,0) - \int_0^1 Q''U(\cdot,0)z(\cdot,0) + \int_0^1 f''z(\cdot,1).$$

Hence

$$\int_\Omega |\nabla z|^2 + \int_0^1 q_m z^2(\cdot,0) \le C_1(\|z(\cdot,0)\|_{L^2(0,1)} + \|z(\cdot,1)\|_{L^2(0,1)}),$$

where C_1 is a positive constant depending on C_0, f and M.

Or the norm $[\int_\Omega |\nabla z|^2 + \int_0^1 q_m z^2(\cdot,0)]^{\frac{1}{2}}$ is equivalent to $\|z\|_{H^1(\Omega)}$ and the trace operator $w \in H^1(\Omega) \to (w(\cdot,0), w(\cdot,1)) \in L^2(0,1)\times L^2(0,1)$ is bounded. Consequently

$$\|z\|_{H^1(\Omega)} \le C_2,$$

for some positive constant C_2 depending on q_m, M and f. This and the continuity of the trace operator $w \in H^1(\Omega) \to w(\cdot,0) \in L^2(0,1)$ imply

$$\|\partial^2_{x^2} U(\cdot,0)\|_{L^2(0,1)} = \|z(\cdot,0)\|_{L^2(0,1)} \le C_3.$$

Here again C_3 is a positive constant depending on q_m, M and f.

The last estimate combined with (4.6) gives

$$\|U(\cdot,0)\|_{H^2(0,1)} \le C_4, \tag{4.7}$$

where C_4 is some positive constant depending on q_m, M and f.

Next let $v = \partial_y u$. Then v is the solution of the following boundary value problem:

$$\begin{cases} \Delta v = 0, & \text{in } \Omega, \\ v(x,0) = q(x)U(x,0), & x \in (0,1), \\ v(x,1) = f(x), & x \in (0,1), \\ \partial_x v(0,y) = \partial_x v(1,y) = 0, & y \in (0,1). \end{cases}$$

We write $v = v_0 + v_1$, where v_0 and v_1 are the respective solutions of the following boundary value problems:

$$\begin{cases} \Delta v_0 = 0, & \text{in } \Omega, \\ v_0(x,0) = q(x)U(x,0), & x \in (0,1), \\ v_0(x,1) = 0, & x \in (0,1), \\ \partial_x v_0(0,y) = \partial_x v_0(1,y) = 0, & y \in (0,1) \end{cases}$$

and

$$\begin{cases} \Delta v_1 = 0, & \text{in } \Omega, \\ v_1(x,0) = 0, & x \in (0,1), \\ v_1(x,1) = f(x), & x \in (0,1), \\ \partial_x v_1(0,y) = \partial_x v_1(1,y) = 0, & y \in (0,1). \end{cases}$$

An application of Green's formula leads to

$$\int_\Omega |\nabla v_0|^2 = -\int_0^1 QU(\cdot,0)\partial_y v_0(\cdot,0).$$

Therefore

$$\begin{aligned} \int_\Omega |\nabla v_0|^2 &\leq \|QU(\cdot,0)\|_{H^{\frac{1}{2}}(0,1)} \|\partial_y v_0(\cdot,0)\|_{H^{-\frac{1}{2}}(0,1)} \\ &\leq A\|Q\|_{W^{1,\infty}(0,1)} \|U(\cdot,0)\|_{H^1(0,1)} \|\partial_y v_0(\cdot,0)\|_{H^{-\frac{1}{2}}(0,1)}, \end{aligned}$$

where A is a positive constant.

On the other hand, we know that the trace operator

$$w \in \{\psi \in H^1(\Omega);\ \Delta\psi \in L^2(\Omega)\} \to \partial_y w(\cdot,0) \in H^{-\frac{1}{2}}(0,1)$$

is bounded. Hence

$$\int_\Omega |\nabla v_0|^2 \leq B\|Q\|_{W^{1,\infty}(0,1)} \|U(\cdot,0)\|_{H^1(0,1)} \|v_0\|_{H^1(\Omega)},$$

for some positive constant B. But $\|\nabla w\|_{L^2(\Omega)}$ is equivalent to $\|w\|_{H^1(\Omega)}$ on $\{\psi \in H^1(\Omega);\ \psi(\cdot,1) = 0\}$. Consequently

$$\|v_0\|_{H^1(\Omega)} \leq D\|Q\|_{W^{1,\infty}(0,1)} \|U(\cdot,0)\|_{H^1(0,1)},$$

where D is a positive constant. Similarly, we have $\|v_1\|_{H^1(\Omega)} \leq E\|f\|_{H^1(0,1)}$ for some positive constant E. Therefore

$$\|v\|_{H^1(\Omega)} \leq F(\|f\|_{H^1(0,1)} + \|Q\|_{W^{1,\infty}(0,1)} \|U(\cdot,0)\|_{H^1(0,1)}),$$

where F is a positive constant, and then

$$\|v(\cdot,0)\|_{L^2(0,1)} \leq G(\|f\|_{H^1(0,1)} + \|Q\|_{W^{1,\infty}(0,1)} \|U(\cdot,0)\|_{H^1(0,1)}), \tag{4.8}$$

from the continuity of the trace operator $w \in H^1(\Omega) \to w(\cdot,0) \in L^2(0,1)$. Here again G is some positive constant.

We now estimate $\|\partial_x v(\cdot,0)\|_{L^2(0,1)}$. We first note that $w = \partial_x v$ is the solution of the boundary value problem

$$\begin{cases} \Delta w = 0. & \text{in } \Omega, \\ w(x,0) = Q'(x)U(x,0) + Q(x)\partial_x U(x,0), & x \in (0,1), \\ w(x,1) = f'(x), & x \in (0,1), \\ w(0,y) = w(1,y) = 0, & y \in (0,1). \end{cases}$$

Proceeding as before we find

$$\|w(\cdot,0)\|_{L^2(0,1)} \le H(\|f'\|_{H^1(0,1)} + \|Q'\|_{W^{1,\infty}}\|U(\cdot,0)\|_{H^1(0,1)} + \|Q\|_{W^{1,\infty}}\|\partial_x U(\cdot,0)\|_{H^1(0,1)}). \tag{4.9}$$

Here H is some positive constant. From a combination of (4.7)–(4.9) we derive

$$\|\partial_y U(\cdot,0)\|_{H^1(0,1)} \le C_5$$

for some positive constant C_5 depending on q_m, f and M.

5. Numerical Example

In this section, we give a numerical example for the inverse problem we discussed in Sec. 4.

We take $\Omega = (0,1)\times(0,1)$, $\Gamma = [0,1]\times\{0\}$, $\gamma = \overline{\partial\Omega\backslash\Gamma}$.

Let us consider the following boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_y u(x,1) + q(x)u(x,1) = 0, & x\in(0,1), \\ \partial_y u(x,0) = f(x), & x\in(0,1), \\ \partial_x u(0,y) = \partial_x u(1,y) = 0, & y\in(0,1). \end{cases} \tag{5.1}$$

Our aim is to determine the coefficient q from Cauchy data f and $g = u(x,0)$.

We can directly verify that the solutions of the problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_x u(0,y) = \partial_x u(1,y) = 0, & y\in(0,1) \end{cases}$$

have the following representation formula:

$$u(x,y) = \sum_{k=0}^{\infty}(c_{k,1}e^{k\pi y} + c_{k,2}e^{-k\pi y})\cos(k\pi x). \tag{5.2}$$

The coefficients $c_{k,1}$ and $c_{k,2}$ can be uniquely determined by the Cauchy data f and g. One can see that the ill-posedness of the inverse problem comes from the factors $e^{k\pi y}$, that is the small errors in the coefficients $c_{k,1}$ will enlarge if k is large.

Our numerical method consists of approximating the solution u by the finite terms in the representation formula (5.2):

$$\tilde{u}(x,y) = \sum_{k=0}^{N}(c_{k,1}e^{k\pi y} + c_{k,2}e^{-k\pi y})\cos(k\pi x).$$

We note that the truncation number N plays the role of a regularization parameter. Then, for any $x\in(0,1)$, the approximation of $q(x)$ can be solved by $\partial_y\tilde{u}(x,1) + q(x)\tilde{u}(x,1) = 0$, i.e.

$$\sum_{k=0}^{N} k\pi(c_{k,1}e^{k\pi} - c_{k,2}e^{-k\pi})\cos(k\pi x) + q_N(x)\left[\sum_{k=0}^{N}(c_{k,1}e^{k\pi} + c_{k,2}e^{-k\pi})\cos(k\pi x)\right] = 0.$$

Numerical example:
Let

$$u(x, y) = 30 + \exp(\pi y)\cos(\pi x).$$

It is easy to verify that the Cauchy data are

$$f(x) = 30 + \cos(\pi x),$$
$$g(x) = \pi \cos(\pi x)$$

and the coefficient is

$$q(x) = -\frac{\pi \exp(\pi)\cos(\pi x)}{30 + \exp(\pi)\cos(\pi x)}.$$

Let N denote the number of terms in the approximating sum. The relationship between N and the errors $\|q - q_N\|_2$ can be found in Fig. 1. We see that the errors $\|q - q_N\|_2$ become larger and larger as N increases.

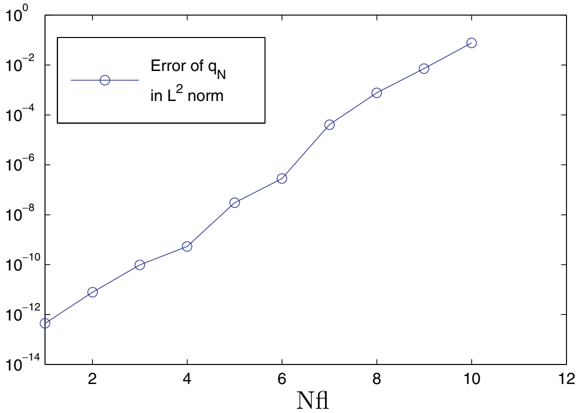

Fig. 1. $\|q - q_N\|_2$ with increasing N.

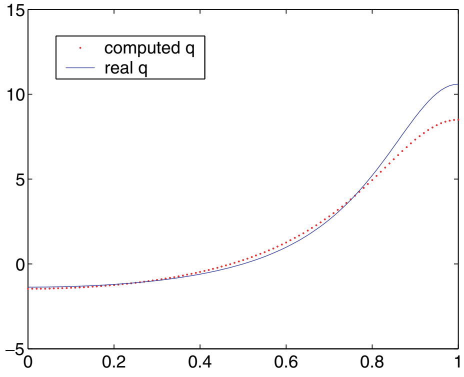

Fig. 2. Numerical solution of q with 2% additive noise on Cauchy data.

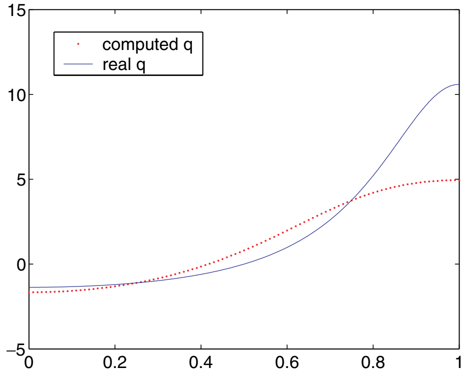

Fig. 3. Numerical solution of q with 5% additive noise on Cauchy data.

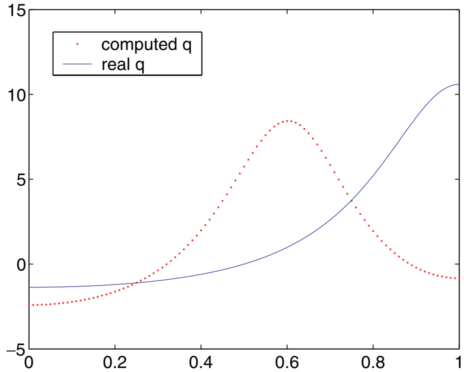

Fig. 4. Numerical solution of q with 10% additive noise on Cauchy data.

We take six terms in the representation formula (5.2). The numerical results with 2%, 5% and 10% noises on Cauchy data are given respectively in Figs. 2–4.

Here the noises we added in data are the random numbers generated by a random number generator. This random number generator can generate the random numbers in $[-\delta, \delta]$, where δ is the noise level.

Remark 6. For the general domain Ω, we are doing the numerical testing by boundary element method. The results will be reported in a forthcoming paper.

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