SIAM J. APPL. MATH.
Vol. 75, No. 5, pp. 2290–2310

ELECTROMAGNETIC FIELD ENHANCEMENT IN SMALL GAPS: A RIGOROUS MATHEMATICAL THEORY*

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Abstract. This paper is concerned with the field enhancement when an electromagnetic wave passes through subwavelength metallic gaps. We focus on a particular configuration when there is extreme scale difference between the wavelength of the incident wave, the thickness of metal films, and the size of gap apertures. Based upon a rigorous study of the perfect electrical conductor model for the transverse magnetic ploarization, we show that enormous electric field enhancement occurs inside the gap if the gap size is sufficiently small. (There is no electromagnetic field enhancement for the transverse electric polarization case, of which the mathematical results are less interesting and they are not presented in this paper.) Furthermore, when the gap size approaches zero, there is a limit on the ultimate enhancement strength, which is proportional to the ratio between the wavelength of the incident wave and the thickness of the metal film. On other hand, it is demonstrated that there is no significant magnetic field enhancement inside the gap.

Key words. electromagnetic field enhancement, nano gap, subwavelength structure, Helmholtz equation

AMS subject classifications. 35Q60, 35Q61, 35J15

DOI. 10.1137/140997890

1. Introduction.

There has been increasing interest in electromagnetic field enhancement and the extraordinary optical transmission effect through subwavelength apertures in recent years, due to its significant potential applications in biological and chemical sensing, spectroscopy, terahertz semiconductor devices, etc. Experimentally it has been shown that when an electromagnetic wave impinges upon a metallic gap, the electric field is significantly enhanced with a decreasing gap width that ranges from micrometers to nanometers. We refer the readers to [4, 8, 9, 13] and references therein for detailed discussions on the experimental investigations. To give an intuition of field enhancement induced by scattering of nanogaps, let us consider a narrow slit formed by two infinitely long metal films that are invariant along the x_3 axis (Figure 1(a)). The slit is in the x_3 direction, and it forms a gap aperture on the x_1x_3 plane with a nanometer gap size. Figure 1(b) shows the enhancement of the electric field in the slit when a polarized terahertz wave impinges normally upon the metal films with different gap sizes. Let E be total electric field after the scattering. Using the incident electrical field E^{inc} as the reference field, it is observed that the quantity $|E|/|E^{inc}|$ keeps increasing inside the slit when the gap width decreases.

A key feature of the configuration in Figure 1 is the extreme scale difference between the wavelength of the incident wave, the thickness of metal films, and the size of gap apertures. In general, the wavelength is in the millimeter regime for terahertz electromagnetic waves that are used in real applications such as terahertz biological and chemical sensing. The thickness of the metal film lies in the range

*Received by the editors December 1, 2014; accepted for publication (in revised form) August 21, 2015; published electronically October 15, 2015.
http://www.siam.org/journals/siap/75-5/99789.html
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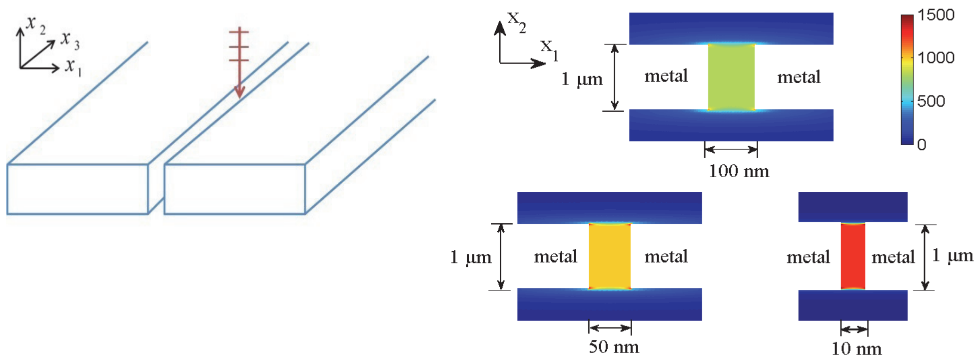

(a) (b)

Fig. 1. *An illustration of the electromagnetic field enhancement.* (a) *nano-slit structure;* (b) *cross-sectional plot of the electric field enhancement factor* $|E|/|E^{inc}|$ *inside the slit for different gap sizes.*

of several hundred nanometers to a few micrometers, while the width of the gap aperture is the nanometer regime. Though enormous field enhancement has been reported both experimentally and numerically [4, 11, 13], there is as yet no theoretical investigation on the strength of the enhancement that occurs in the slit and the ultimate enhancement factor that can be achieved as the gap size approaches zero. The goal of this paper is to provide a quantitative analysis of the electromagnetic field enhancement for such configurations based on a rigorous study of the underlying mathematical model in the classical regime. In particular, we point out that if the gap size is sufficiently small, the enhancement factor for the electric field in the case of transverse magnetic (TM) polarization is proportional to the ratio between the wavelength of the incident wave and the thickness of the metal film. The strength could exceed 10,000 due to the scale difference between the two in real applications.

Let us begin with the description of the geometry for the problem under consideration. Throughout the paper, we adopt the notation $x = (x_1, x_2)$. The metallic structure is invariant along x_3 and its shape on the x_1x_2 plane is depicted in Figure 2. The metal films have a thickness of l. In general, the metal is long in the x_1 direction. For the sake of simplicity, here we assume that each metal film is semi-infinite along x_1 as shown in Figure 2. An infinitely long slit is formed, which has a rectangular cross section S_ε of width ε on the x_1x_2 plane. In the extreme case when $\varepsilon = 0$, two metal films become an infinite slab of thickness l. Let ω be the operating frequency of the electromagnetic wave and c be the speed of the wave in the vacuum. We consider a polarized time-harmonic electromagnetic wave (with $e^{-i\omega t}$ dependence) that impinges upon the metal films, wherein the magnetic field vector is parallel to the x_3 axis such that the incident wave $H^i = (0, 0, u^i)$. Here $u^i = e^{ikd\cdot x}$ is a plane wave propagating along the direction d, and k is the wavenumber defined as $k = \frac{\omega}{c}$. The corresponding incident electric field E^i is determined by the Ampere's law $\nabla \times H^i = -i\omega\tau_0 E^i$, wherein τ_0 is the electrical permittivity in the vacuum. Let us denote the wavelength of the incident wave by λ. As discussed above, in this paper we are interested in the case when the length scale of the underlying geometry is given by $\varepsilon \ll l \ll \lambda$ such that significant field enhancement as shown in Figure 1 occurs.

We restrict our discussion to the x_1x_2 plane since the problem under consideration is the TM polarized case. The standard Cartesian coordinate system is adopted such

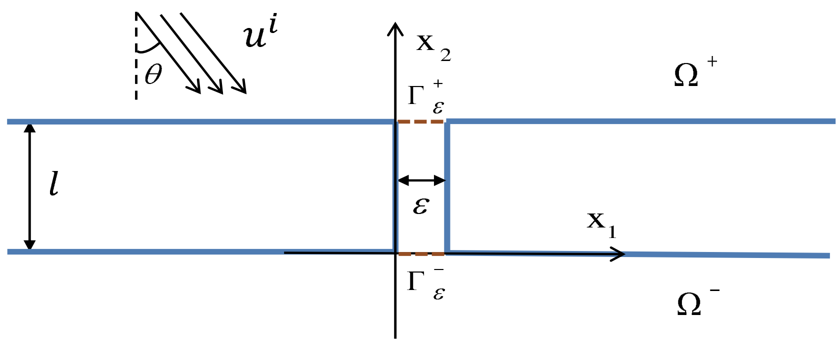

FIG. 2. *Geometry of the problem. The slit S_ε has a rectangular shape of length l and width ε, respectively. The domains above and below the metals films are denoted as Ω_+ and Ω_-, respectively, and the domain exterior to the metal films is denoted as Ω_ε, which consists of S_ε, Ω_+, and Ω_-.*

that the origin corresponds to the lower left corner of the slit. Hence, the slit region $S_\varepsilon = \{(x_1, x_2) \mid 0 < x_1 < \varepsilon, 0 < x_2 < l\}$. We denote the upper and lower gap apertures by $\Gamma_{+,\varepsilon}$ and $\Gamma_{-,\varepsilon}$, respectively, and denote the semi-infinite domain above and below the metal films by Ω_+ and Ω_-, respectively (see Figure 2). Let Ω_ε be the domain exterior to the metal films. It is seen that $\Omega_\varepsilon = S_\varepsilon \cup \Omega_+ \cup \Omega_-$. Let θ be the incident angle such that $0 \le \theta < \frac{\pi}{2}$; then the propagation direction for the incident wave takes the form $d = (\sin\theta,\ -\cos\theta)^T$. Let u^r be the reflected field by the metal plane $\{x_2 = l\}$ (in the absence of slit). It can be shown that $u^r = e^{-i2k\cos\theta l} e^{ikd'\cdot x}$ and it propagates along the direction $d' = (\sin\theta,\ \cos\theta)^T$. In general, with the slit S_ε, the total field u_ε above the metal films after the scattering consists of three parts: the incident wave u^i, the reflected wave u^r, and the scattered field u_ε^s. Below the metal films, the transmitted wave $u_\varepsilon = u_\varepsilon^s$. The total field u_ε satisfies the Helmholtz equation

$$\Delta u_\varepsilon + k^2 u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon. \tag{1.1}$$

The metal is a perfect electrical conductor (PEC) and the wavelength considered in this paper is in the millimeter regime; thus we have the boundary condition

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{1.2}$$

In addition, at infinity the scattered field u_ε^s satisfies the Sommerfeld radiation condition (cf. [6]):

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s\right) = 0, \quad r = |x|. \tag{1.3}$$

We define the electromagnetic field enhancement factor in the slit S_ε as the ratio between the electromagnetic energy in this region when the metal films are present and when they are not present. It is observed that in the absence of metal films, the electromagnetic fields in the slit region are simply incident fields $\{E^i, H^i\}$. Let $\{E_\varepsilon, H_\varepsilon\}$ be the total electromagnetic fields due to the scattering by the slit and metal films. Then, using the incident wave $\{E^i, H^i\}$ as the reference fields, the electric and magnetic field enhancement factors inside the slit are given by

$$Q_E = \frac{||E_\varepsilon||_{L^2(S_\varepsilon)}}{||E^i||_{L^2(S_\varepsilon)}} \quad \text{and} \quad Q_H = \frac{||H_\varepsilon||_{L^2(S_\varepsilon)}}{||H^i||_{L^2(S_\varepsilon)}},$$

respectively. Our main results regarding the electromagnetic field enhancement factor are stated as follows.

THEOREM 1.1. *Assume that $l \ll \lambda$. Let E_ε be the total electric field due to the scattering by the slit S_ε, and let E^i be the incident electric field. Then there exists a positive constant ε_0 depending on l and λ such that for all $0 < \varepsilon < \varepsilon_0$, the following estimate holds:*

$$C_1 \left(\frac{\lambda}{l}\right) \leq \frac{||E_\varepsilon||_{L^2(S_\varepsilon)}}{||E^i||_{L^2(S_\varepsilon)}} \leq C_2 \left(\frac{\lambda}{l}\right).$$

C_1 and C_2 are some positive constants independent of ε_0, ε, l, and λ.

THEOREM 1.2. *Assume that $l \ll \lambda$. Let H_ε be the total magnetic field due to the scattering by the slit S_ε, and let H^i be the incident electric field. Then*

$$C_1 \leq \frac{||H_\varepsilon||_{L^2(S_\varepsilon)}}{||H^i||_{L^2(S_\varepsilon)}} \leq C_2$$

for $0 < \varepsilon < \varepsilon_0$, where the constant ε_0 depends on l and λ, and C_1, C_2 are independent of ε_0, ε, l, and λ.

From Theorem 1.1, we deduce that electric field enhancement with an order of $O(\lambda/l)$ occurs inside the slit due to the scattering by the nanogap. Note that the length scale $l \ll \lambda$ is prescribed for the geometry, and thus the enhancement strength is enormous for such a configuration when the gap size is sufficiently small. Furthermore, Theorem 1.1 provides a limit on the ultimate enhancement factor as $\varepsilon \to 0$. On the other hand, no significant magnetic field enhancement is gained from the scattering of the slit when the gap size shrinks, as indicated by Theorem 1.2. The readers are also referred to [11] for a numerical study of such phenomena. Due to the scaling invariance of the Maxwell equations, it follows that such enhancement behavior remains true in any electromagnetic frequency regime, as long as $l \ll \lambda$ and the gap size ε reaches a sufficiently small threshold. We point out that our work is significantly different from the semi-infinite slit discussed in [12] or the finite slit in [5]. This paper focuses on the geometry with a particular length scale so that the enhancement is huge inside the slit, while no enhancement was reported in [5, 12]. The readers are also referred to [14, 15, 16] for the study of the enhancement for thick metals due to the mechanism of Fabry–Pérot resonances.

The rest of the paper is organized as follows. Section 2 introduces an approximate model for the problem (1.1)–(1.3). The study of the electromagnetic field enhancement and the proof of Theorems 1.1 and 1.2 are given in section 3. The mathematical theory is based on the investigation for the solution of the approximate model and an estimate for the error of such an approximation, where the latter is established in section 4. The paper is concluded with some general remarks in section 5.

2. An approximate model for wave scattering by the slit.

In order to estimate the field enhancement inside the slit, we introduce an approximate model for the problem (1.1)–(1.3). To begin with, we expand the wave field u_ε inside the slit S_ε as the sum of wave-guide modes:

$$u_\varepsilon(x) = \sum_{n=0}^{\infty} \left(\alpha_n^- e^{i\gamma_n x_2} + \alpha_n^+ e^{-i\gamma_n (x_2 - l)}\right) \phi_n(x_1), \qquad (2.1)$$

wherein $\phi_0(x_1) = \frac{1}{\sqrt{\varepsilon}}$ and $\phi_n(x_1) = \sqrt{\frac{2}{\varepsilon}} \cos(\frac{n\pi x_1}{\varepsilon})$ (for $n \geq 1$) form an orthonormal basis on the interval $[0, \varepsilon]$. The coefficients γ_n are defined as

$$\gamma_n = \begin{cases} k, & n = 0; \\ i\,[(n\pi/\varepsilon)^2 - k^2]^{1/2}, & n \geq 1, \end{cases}$$

wherein γ_n are pure imaginary numbers for $n \geq 1$ if ε is sufficiently small. It is noted that for each n, the expansion consists of two modes propagating upward and downward, respectively, if $n\pi/\varepsilon \leq k$, and decaying exponentially away from lower and upper gap apertures, respectively, if $n\pi/\varepsilon > k$. The mathematical model (1.1)–(1.3) may be reformulated as the following equivalent coupled problem using the expansion (2.1):

$$(2.2)\quad \begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \qquad \text{in } \Omega^+ \cup \Omega^-, \\ u_\varepsilon(x) = \displaystyle\sum_{n=0}^{\infty} \left(\alpha_n^- e^{i\gamma_n x_2} + \alpha_n^+ e^{-i\gamma_n (x_2 - l)} \right) \phi_n(x_1) \qquad \text{in } S_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 \qquad \text{on } (\{x_2 = l\} \backslash \Gamma_\varepsilon^+) \cup (\{x_2 = 0\} \backslash \Gamma_\varepsilon^-), \\ u_\varepsilon(x_1, l+) = u_\varepsilon(x_1, l-), \qquad u_\varepsilon(x_1, 0+) = u_\varepsilon(x_1, 0-), \quad x_1 \in (0, \varepsilon), \\ \dfrac{\partial u_\varepsilon}{\partial n}(x_1, l+) = \dfrac{\partial u_\varepsilon}{\partial n}(x_1, l-), \quad \dfrac{\partial u_\varepsilon}{\partial n}(x_1, 0+) = \dfrac{\partial u_\varepsilon}{\partial n}(x_1, 0-), \quad x_1 \in (0, \varepsilon), \\ \displaystyle\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s \right) = 0, \\ \text{wherein } u_\varepsilon^s = u_\varepsilon - (u^i + u^r) \text{ in } \Omega^+ \text{ and } u_\varepsilon^s = u_\varepsilon \text{ in } \Omega^-. \end{cases}$$

The continuity of the wave fields are imposed along the gap apertures, wherein + and − indicate the limit taken from above and below the apertures, respectively.

For simplicity of notation, we define $u_\varepsilon^+(x_1) = u_\varepsilon(x_1, l)$ and $u_\varepsilon^-(x_1) = u_\varepsilon(x_1, 0)$. Let $u_{\varepsilon,n}^+ = \langle u_\varepsilon^+, \phi_n \rangle$ and $u_{\varepsilon,n}^- = \langle u_\varepsilon^-, \phi_n \rangle$ be the Fourier coefficients for u_ε^+ and u_ε^-, respectively, where the inner product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle u_\varepsilon^\pm, \phi_n \rangle := \int_0^\varepsilon u_\varepsilon^\pm(x_1) \phi_n(x_1)\, dx_1.$$

Let $\Gamma_\varepsilon = \Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-$, and $\mathbf{u}_\varepsilon = [u_\varepsilon^+, u_\varepsilon^-]$. We introduce the Sobolev space $H^s(\Gamma_\varepsilon)$ with the norm

$$(2.3)\qquad \|\mathbf{u}_\varepsilon\|_{H^s(\Gamma_\varepsilon)}^2 = \sum_{n=0}^{\infty} \left(1 + \left(\frac{n\pi}{\varepsilon} \right)^2 \right)^s \left(|u_{\varepsilon,n}^+|^2 + |u_{\varepsilon,n}^-|^2 \right).$$

In addition, we define the semi-norm $|\mathbf{u}|_{H^s(\Gamma_\varepsilon)}$ on $H^s(\Gamma_\varepsilon)$ by letting

$$(2.4)\qquad |\mathbf{u}_\varepsilon|_{H^s(\Gamma_\varepsilon)}^2 = \sum_{n=1}^{\infty} \left(1 + \left(\frac{n\pi}{\varepsilon} \right)^2 \right)^s \left(|u_{\varepsilon,n}^+|^2 + |u_{\varepsilon,n}^-|^2 \right).$$

By matching the expansion series (2.1) over the gap apertures and using the continuity conditions prescribed in (2.2), it can be shown that the expansion coefficients are given by

$$(2.5)\qquad \alpha_n^- = \frac{e^{i\gamma_n l} u_{\varepsilon,n}^+ - u_{\varepsilon,n}^-}{e^{i2\gamma_n l} - 1} \quad \text{and} \quad \alpha_n^+ = \frac{e^{i\gamma_n l} u_{\varepsilon,n}^- - u_{\varepsilon,n}^+}{e^{i2\gamma_n l} - 1}.$$

Therefore, in the slit S_ε, the wave field

$$u_\varepsilon(x) = \sum_{n=0}^{\infty} \left(\frac{e^{i\gamma_n l} u^+_{\varepsilon,n} - u^-_{\varepsilon,n}}{e^{i2\gamma_n l} - 1} e^{i\gamma_n x_2} + \frac{e^{i\gamma_n l} u^-_{\varepsilon,n} - u^+_{\varepsilon,n}}{e^{i2\gamma_n l} - 1} e^{-i\gamma_n (x_2 - l)} \right) \phi_n(x_1). \tag{2.6}$$

A direct calculation yields the normal derivatives on Γ^+_ε and Γ^-_ε, which are expressed as

$$\frac{\partial u_\varepsilon(x_1, l-)}{\partial n} = \sum_{n=0}^{\infty} i\gamma_n \left(\frac{e^{i\gamma_n l} u^+_{\varepsilon,n} - u^-_{\varepsilon,n}}{e^{i2\gamma_n l} - 1} e^{i\gamma_n l} - \frac{e^{i\gamma_n l} u^-_{\varepsilon,n} - u^+_{\varepsilon,n}}{e^{i2\gamma_n l} - 1} \right) \phi_n(x_1) \quad \text{on } \Gamma^+_\varepsilon \tag{2.7}$$

and

$$\frac{\partial u_\varepsilon(x_1, 0+)}{\partial n} = \sum_{n=0}^{\infty} i\gamma_n \left(-\frac{e^{i\gamma_n l} u^+_{\varepsilon,n} - u^-_{\varepsilon,n}}{e^{i2\gamma_n l} - 1} + \frac{e^{i\gamma_n l} u^-_{\varepsilon,n} - u^+_{\varepsilon,n}}{e^{i2\gamma_n l} - 1} e^{i\gamma_n l} \right) \phi_n(x_1) \quad \text{on } \Gamma^-_\varepsilon. \tag{2.8}$$

Here n is the unit normal on Γ^+_ε and Γ^-_ε pointing into the exterior domain Ω^+ and Ω^-, respectively.

For a function $\psi \in H^1(\Omega^+ \cup \Omega^-)$, following the notation above, we set $\psi^+ = \psi(x_1, l)$ and $\psi^- = \psi(x_1, 0)$. For a given vector function $\boldsymbol{\psi} = [\psi^+, \psi^-]$, let us define the Dirichlet-to-Neumann map on Γ^+_ε and Γ^-_ε as

$$\Lambda^+_\varepsilon(\boldsymbol{\psi}) = \sum_{n=0}^{\infty} i\gamma_n \left(\frac{e^{i\gamma_n l} \psi^+_n - \psi^-_n}{e^{i2\gamma_n l} - 1} e^{i\gamma_n l} - \frac{e^{i\gamma_n l} \psi^-_n - \psi^+_n}{e^{i2\gamma_n l} - 1} \right) \phi_n(x_1) \tag{2.9}$$

and

$$\Lambda^-_\varepsilon(\boldsymbol{\psi}) = \sum_{n=0}^{\infty} i\gamma_n \left(-\frac{e^{i\gamma_n l} \psi^+_n - \psi^-_n}{e^{i2\gamma_n l} - 1} + \frac{e^{i\gamma_n l} \psi^-_n - \psi^+_n}{e^{i2\gamma_n l} - 1} e^{i\gamma_n l} \right) \phi_n(x_1), \tag{2.10}$$

where the Fourier coefficients $\psi^+_n = \langle \psi^+, \phi_n \rangle$ and $\psi^-_n = \langle \psi^-, \phi_n \rangle$. Define the operator

$$\Lambda_\varepsilon : \boldsymbol{\psi} \to [\Lambda^+_\varepsilon(\boldsymbol{\psi}), \Lambda^-_\varepsilon(\boldsymbol{\psi})]. \tag{2.11}$$

LEMMA 2.1. *The Dirichlet-to-Neumann map Λ_ε defined by* (2.11) *is bounded from $H^{\frac{1}{2}}(\Gamma_\varepsilon)$ to $H^{-\frac{1}{2}}(\Gamma_\varepsilon)$ if $\varepsilon \ll l \ll \lambda$.*

Proof. For any $\boldsymbol{\psi}$, $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma_\varepsilon)$,

$$\langle \Lambda_\varepsilon \boldsymbol{\psi}, \boldsymbol{\varphi} \rangle = \sum_{n=0}^{\infty} i\gamma_n \left(e^{i\gamma_n l} c^-_n - c^+_n \right) \varphi^+_n + i\gamma_n \left(-c^-_n + e^{i\gamma_n l} c^+_n \right) \varphi^-_n,$$

where $c^-_n = \frac{e^{i\gamma_n l}\psi^+_n - \psi^-_n}{e^{i2\gamma_n l} - 1}$ and $c^+_n = \frac{e^{i\gamma_n l}\psi^-_n - \psi^+_n}{e^{i2\gamma_n l} - 1}$. Note that the coefficients γ_n are given by

$$\gamma_n = \begin{cases} k, & n = 0; \\ i\,[(n\pi/\varepsilon)^2 - k^2]^{1/2}, & n \geq 1. \end{cases}$$

A straightforward calculation yields

$$\begin{aligned} &\left| i\gamma_0 \left(e^{i\gamma_0 l} c^-_0 - c^+_0 \right) \varphi^+_0 + i\gamma_0 \left(-c^-_0 + e^{i\gamma_0 l} c^+_0 \right) \varphi^-_0 \right| \\ &\quad \leq \frac{k}{|\sin(kl)|} \left(|\psi^+_0| + |\psi^-_0| \right) \left(|\varphi^+_0| + |\varphi^-_0| \right). \end{aligned}$$

For $n \geq 1$, using the fact $\varepsilon \ll l \ll \lambda$, it is obtained that

$$|i\gamma_n \left(e^{i\gamma_n l} c_n^- - c_n^+\right) \varphi_n^+ + i\gamma_n \left(-c_n^- + e^{i\gamma_n l} c_n^+\right) \varphi_n^-| \leq \frac{\tilde{c} n\pi}{\varepsilon} \left(|\psi_n^+| + |\psi_n^-|\right) \left(|\varphi_n^+| + |\varphi_n^-|\right)$$

for some constant $\tilde{c}$. A combination of the above two inequalities and the Cauchy–Schwartz inequality leads to

$$|\langle \Lambda_\varepsilon \boldsymbol{\psi}, \boldsymbol{\varphi} \rangle| \leq C(k,l) \, \|\boldsymbol{\psi}\|_{H^{1/2}(\Gamma_\varepsilon)} \, \|\boldsymbol{\varphi}\|_{H^{1/2}(\Gamma_\varepsilon)} \, .$$

Hence, the boundness of Λ_ε follows. ∎

By virtue of (2.7)–(2.10) and the continuity of $\frac{\partial u_\varepsilon}{\partial n}$ at gap apertures as described in (2.2), we obtain the following boundary conditions for u_ε on Γ_ε^+ and Γ_ε^-:

$$\begin{bmatrix} \partial u_\varepsilon(x_1, l)/\partial n \\ \partial u_\varepsilon(x_1, 0)/\partial n \end{bmatrix} = \begin{bmatrix} \Lambda_\varepsilon^+(\mathbf{u}_\varepsilon) \\ \Lambda_\varepsilon^-(\mathbf{u}_\varepsilon) \end{bmatrix}. \tag{2.12}$$

With the boundary condition defined above, it is seen that (1.1)–(1.3) can be reduced to a boundary value problem in the domain $\Omega^+ \cup \Omega^-$ as follows:

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \qquad \text{in } \Omega^+ \cup \Omega^-, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 \qquad \text{on } (\{x_2 = l\} \cup \{x_2 = 0\}) \setminus (\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-), \\ \dfrac{\partial u_\varepsilon}{\partial n} = \Lambda_\varepsilon^\pm(\mathbf{u}_\varepsilon) \quad \text{on } \Gamma_\varepsilon^\pm, \\ \lim\limits_{r \to \infty} \sqrt{r} \left(\dfrac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s \right) = 0, \\ \text{wherein } u_\varepsilon^s = u_\varepsilon - (u^i + u^r) \text{ in } \Omega^+ \text{ and } u_\varepsilon^s = u_\varepsilon \text{ in } \Omega^-. \end{cases} \tag{2.13}$$

Finally, by calculating the Fourier coefficients $u_{\varepsilon,n}^+$ and $u_{\varepsilon,n}^-$, the solution in the slit S_ε may be obtained by the formula (2.6).

To approximate the model (2.13), we examine the expansion series (2.1) closely. Note that if ε is sufficiently small such that $\varepsilon/\lambda \ll 1$, the modes $e^{i\gamma_n x_2}\phi_n(x_1)$ and $e^{-i\gamma_n(x_2 - l)}\phi_n(x_1)$ decay exponentially away from the lower and upper gap apertures, respectively, with a rate of $O(e^{-\frac{n}{\varepsilon}})$ for all $n \geq 1$. That is, only the dominant modes $e^{ikx_2}\phi_0(x_1)$ and $e^{ik(l-x_2)}\phi_0(x_1)$ propagate through the slit. This crucial fact is also observed numerically in [11]. The observation motivates us to approximate the Dirichlet-to-Neumann map (2.9)–(2.10) by dropping off the high order modes or, equivalently, by using the operators $\tilde{\Lambda}_\varepsilon^+(\boldsymbol{\psi})$ and $\tilde{\Lambda}_\varepsilon^-(\boldsymbol{\psi})$ defined by

$$\tilde{\Lambda}_\varepsilon^+(\boldsymbol{\psi}) = ik \left(\frac{e^{ikl}\psi_0^+ - \psi_0^-}{e^{i2kl} - 1} e^{ikl} - \frac{e^{ikl}\psi_0^- - \psi_0^+}{e^{i2kl} - 1} \right) \phi_0(x_1) \tag{2.14}$$

and

$$\tilde{\Lambda}_\varepsilon^-(\boldsymbol{\psi}) = ik \left(-\frac{e^{ikl}\psi_0^+ - \psi_0^-}{e^{i2kl} - 1} + \frac{e^{ikl}\psi_0^- - \psi_0^+}{e^{i2kl} - 1} e^{ikl} \right) \phi_0(x_1). \tag{2.15}$$

In this way, we approximate the boundary value problem (2.13) using the following model defined over the domain $\Omega^+ \cup \Omega^-$:

$$\text{(2.16)} \qquad \begin{cases} \Delta v_\varepsilon + k^2 v_\varepsilon = 0 \qquad \text{in } \Omega^+ \cup \Omega^-, \\ \dfrac{\partial v_\varepsilon}{\partial n} = 0 \qquad \text{on } \ (\{x_2 = l\} \cup \{x_2 = 0\}) \setminus (\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-), \\ \dfrac{\partial v_\varepsilon}{\partial n} = \tilde{\Lambda}_\varepsilon^\pm(\mathbf{v}_\varepsilon) \quad \text{on } \Gamma_\varepsilon^\pm. \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial v_\varepsilon^s}{\partial r} - ikv_\varepsilon^s\right) = 0, \\ \text{wherein } v_\varepsilon^s = v_\varepsilon - (u^i + u^r) \text{ in } \Omega^+ \text{ and } v_\varepsilon^s = v_\varepsilon \text{ in } \Omega^-. \end{cases}$$

We point out that a similar approximate model has also been proposed for the study of semi-infinite slits in [12]. Accordingly, inside the slit S_ε, we approximate u_ε with one single mode:

$$\text{(2.17)} \qquad v_\varepsilon(x) = \left(\tilde{\alpha}_0^- e^{ikx_2} + \tilde{\alpha}_0^+ e^{-ik(x_2-l)}\right)\phi_0(x_1),$$

where the coefficients $\tilde{\alpha}_0^-$ and $\tilde{\alpha}_0^+$ are given by

$$\text{(2.18)} \qquad \tilde{\alpha}_0^- = \frac{e^{ikl}v_{\varepsilon,0}^+ - v_{\varepsilon,0}^-}{e^{i2kl} - 1} \quad \text{and} \quad \tilde{\alpha}_0^+ = \frac{e^{ikl}v_{\varepsilon,0}^- - v_{\varepsilon,0}^+}{e^{i2kl} - 1}.$$

3. Electromagnetic field enhancement inside the slit.

We explore the electromagnetic field enhancement inside the slit due to the scattering by the small gap. The study of the field enhancement is based on estimates for the solution to the approximate model (2.16)–(2.18) and estimates for the error of the approximate solution, which is discussed in sections 3.1 and 3.2, respectively. The proof for the electromagnetic field enhancement, namely, Theorems 1.1 and 1.2, is given in section 3.3.

3.1. Approximate field inside the slit.

THEOREM 3.1. *Assume that $l \ll \lambda$. Let v_ε be the solution of the approximate model* (2.16)*, and let v_ε be given by* (2.17) *inside the silt; then there exist positive constants C_1 and C_2 independent of ε, l, and λ such that*

$$C_1\sqrt{\frac{\varepsilon}{l}} \le ||\nabla v_\varepsilon||_{L^2(S_\varepsilon)} \le C_2\sqrt{\frac{\varepsilon}{l}}$$

for sufficiently small ε.

Proof. Define the half-space Green's function $G(x,y) = -\frac{i}{4}(H_0^{(1)}(k|x-y|) + H_0^{(1)}(k|x'-y|))$, where $H_0^{(1)}$ is the first kind Hankel function of order 0 [1], and x' is the reflection of the point x with respect to the line $\{x_2 = l\}$. Let $v_\varepsilon(x)$ be the solution of (2.16). Using the Green's identity and noting that $\frac{\partial u^i}{\partial n} + \frac{\partial u^r}{\partial n} = 0$ on $\{x_2 = l\}$, it follows that

$$v_\varepsilon^s(x) = \int_{\Gamma_\varepsilon^+} G(x,y)\frac{\partial v_\varepsilon^s}{\partial n_y}ds_y, \quad x \in \Omega^+,$$

and

$$v_\varepsilon(x) = u^i + u^r + \int_{\Gamma_\varepsilon^+} G(x,y)\frac{\partial v_\varepsilon}{\partial n_y}ds_y, \quad x \in \Omega^+.$$

Here n is the unit normal on Γ_ε^+ pointing into the exterior domain Ω^+. From the continuity of the single layer potential [6, 7], it is seen that

$$v_\varepsilon(x) = u^i + u^r - \frac{i}{2}\int_{\Gamma_\varepsilon^+} H_0^{(1)}(k|x-y|)\frac{\partial v_\varepsilon}{\partial n_y} ds_y \quad \text{for } x \in \Gamma_\varepsilon^+. \tag{3.1}$$

Similarly, it can be obtained that

$$v_\varepsilon(x) = -\frac{i}{2}\int_{\Gamma_\varepsilon^-} H_0^{(1)}(k|x-y|)\frac{\partial v_\varepsilon}{\partial n_y} ds_y \quad \text{for } x \in \Gamma_\varepsilon^-. \tag{3.2}$$

In light of (2.14)–(2.16), and (2.18), it follows that

$$\frac{\partial v_\varepsilon}{\partial n} = ik(\tilde{\alpha}_0^- e^{ikl} - \tilde{\alpha}_0^+)\phi_0(y_1) \quad \text{on } \Gamma_\varepsilon^+, \quad \text{and} \quad \frac{\partial v_\varepsilon}{\partial n} = ik(-\tilde{\alpha}_0^- + \tilde{\alpha}_0^+ e^{ikl})\phi_0(y_1) \ \text{ on } \Gamma_\varepsilon^-.$$

By substituting into (3.1) and (3.2), and using the fact that $\phi_0(x_1) = \frac{1}{\sqrt{\varepsilon}}$, it is obtained that

$$v_\varepsilon(x_1, l) = u^i(x_1, l) + u^r(x_1, l) + \frac{k}{2}(\tilde{\alpha}_0^- e^{ikl} - \tilde{\alpha}_0^+)\ \frac{1}{\sqrt{\varepsilon}}\ h(x_1), \qquad x_1 \in (0, \varepsilon),$$
$$v_\varepsilon(x_1, 0) = \frac{k}{2}(\tilde{\alpha}_0^+ e^{ikl} - \tilde{\alpha}_0^-)\ \frac{1}{\sqrt{\varepsilon}}\ h(x_1), \qquad x_1 \in (0, \varepsilon).$$

In the above equations, we have adopted the notation

$$h(x_1) = \int_0^\varepsilon H_0^{(1)}(k|x_1 - y_1|)dy_1.$$

Therefore, the Fourier coefficients $v_{\varepsilon,0}^+$ and $v_{\varepsilon,0}^-$ may be expressed as

$$v_{\varepsilon,0}^+ = \langle u^i(\cdot, l), \phi_0\rangle + \langle u^r(\cdot, l), \phi_0\rangle + \frac{k}{2}(\tilde{\alpha}_0^- e^{ikl} - \tilde{\alpha}_0^+)\ \frac{1}{\sqrt{\varepsilon}}\ \ \langle h, \phi_0\rangle,$$
$$v_{\varepsilon,0}^- = \frac{k}{2}(\tilde{\alpha}_0^+ e^{ikl} - \tilde{\alpha}_0^-)\ \frac{1}{\sqrt{\varepsilon}}\ \ \langle h, \phi_0\rangle.$$

From (2.17) and the fact that $u^i(x_1, l) = u^r(x_1, l)$, it yields

$$\tilde{\alpha}_0^- e^{ikl} + \tilde{\alpha}_0^+ = 2\langle u^i(\cdot, l), \phi_0\rangle + \frac{k}{2\sqrt{\varepsilon}}\ \langle h, \phi_0\rangle\ (\tilde{\alpha}_0^- e^{ikl} - \tilde{\alpha}_0^+),$$
$$\tilde{\alpha}_0^- + \tilde{\alpha}_0^+ e^{ikl} = \frac{k}{2\sqrt{\varepsilon}}\ \langle h, \phi_0\rangle\ (\tilde{\alpha}_0^+ e^{ikl} - \tilde{\alpha}_0^-).$$

The coefficients $\tilde{\alpha}_0^+$ and $\tilde{\alpha}_0^-$ can now be obtained as follows by solving the above linear system:

$$\tilde{\alpha}_0^- = e^{ikl}\frac{2\langle u^i(\cdot, l), \phi_0\rangle(1 - c_0)}{e^{i2kl}(1 - c_0)^2 - (1 + c_0)^2} \quad \text{and} \quad \tilde{\alpha}_0^+ = \frac{-2\langle u^i(\cdot, l), \phi_0\rangle(1 + c_0)}{e^{i2kl}(1 - c_0)^2 - (1 + c_0)^2}, \tag{3.3}$$

where $c_0 = \dfrac{k}{2\sqrt{\varepsilon}}\ \langle h, \phi_0\rangle$.

Note that for small $|x_1 - y_1|$, asymptotically, it holds that (cf. [1])

$$H_0^{(1)}(k|x_1 - y_1|) = \frac{2i}{\pi}\ln|x_1 - y_1| + \frac{2i}{\pi}\ln\frac{k}{2} + C + O(|x_1 - y_1|^2 \ln|x_1 - y_1|),$$

wherein the constant C is the Euler's constant defined by $\lim_{p\to\infty}\sum_{m=1}^{p}\dfrac{1}{m}-\ln p$. A direct calculation yields

$$c_0=\frac{k}{2\sqrt{\varepsilon}}\,\langle h,\phi_0\rangle=\frac{ik}{\pi}\varepsilon\ln\varepsilon+O(\varepsilon^2). \tag{3.4}$$

Using the fact that $l\ll\lambda$, for sufficiently small ε, it is obtained that

$$|e^{i2kl}(1-c_0)^2-(1+c_0)^2|\le 2|e^{i2kl}-1|\le 8kl, \tag{3.5}$$

$$|e^{i2kl}(1-c_0)^2-(1+c_0)^2|\ge \frac{1}{2}|e^{i2kl}-1|\ge \frac{1}{2}kl, \tag{3.6}$$

and

$$\frac{1}{2}\le|1+c_0|\le 2,\quad \frac{1}{2}\le|1-c_0|\le 2. \tag{3.7}$$

On the other hand, $|\langle u^i(\cdot,l),\phi_0\rangle|\le\sqrt{\varepsilon}$. Therefore, by combining (3.5)–(3.7), we have

$$|\tilde{\alpha}_0^-|\le C\frac{\sqrt{\varepsilon}}{kl}\quad\text{and}\quad|\tilde{\alpha}_0^+|\le C\frac{\sqrt{\varepsilon}}{kl}.$$

From the expression (2.17) for $v_\varepsilon(x)$ inside the slit, we see that

$$||\nabla v_\varepsilon||^2_{L^2(S_\varepsilon)}\le\int_0^\varepsilon\int_0^l\left(k(|\tilde{\alpha}_0^-|+|\tilde{\alpha}_0^+|)\frac{1}{\sqrt{\varepsilon}}\right)^2dx_2dx_1\le C_2^2\frac{\varepsilon}{l}$$

for some constant C_2.

Substituting (3.3) into (2.17), we get

$$\frac{\partial v_\varepsilon}{\partial x_2}=ike^{ikl}\frac{2\langle u^i(\cdot,l),\phi_0\rangle}{e^{i2kl}(1-c_0)^2-(1+c_0)^2}\left((1-c_0)e^{ikx_2}+(1+c_0)e^{-ikx_2}\right)\phi_0(x_1).$$

By virtue of (3.5) and a straightforward calculation, it follows that, if ε is sufficiently small, there exists a positive constant C such that

$$\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|\ge C\frac{\sqrt{\varepsilon}}{l}\;\;|2\cos(kx_2)-2ic_0\sin(kx_2)|\,|\phi_0(x_1)|\ge C\,|\cos(kx_2)|\;\;\frac{1}{l}.$$

Note that $kl\ll1$, thus $\int_0^l\int_0^\varepsilon|\cos(kx_2)|^2dx_1dx_2\ge\frac{\varepsilon l}{2}$, and we obtain the desired lower bound

$$||\nabla v_\varepsilon||^2_{L^2(S_\varepsilon)}\ge C_1^2\frac{\varepsilon}{l}$$

for some constant C_1. This completes the proof. ◻

THEOREM 3.2. *Assume that $l\ll\lambda$. Let v_ε be the solution of the approximate model* (2.16), *and let v_ε be given by* (2.17); *then there exist positive constants C_1 and C_2 independent of ε, l, and λ such that*

$$C_1\sqrt{\varepsilon l}\le||v_\varepsilon||_{L^2(S_\varepsilon)}\le C_2\sqrt{\varepsilon l}$$

for sufficiently small ε.

Proof. By substituting (3.3) into (2.17), it follows that

$$\begin{aligned} v_\varepsilon(x_1,x_2) &= e^{ikl}\frac{2\langle u^i(\cdot,l),\phi_0\rangle}{e^{i2kl}(1-c_0)^2-(1+c_0)^2}\left((1-c_0)e^{ikx_2}-(1+c_0)e^{-ikx_2}\right)\phi_0(x_1) \\ &= e^{ikl}\frac{2\langle u^i(\cdot,l),\phi_0\rangle}{e^{i2kl}(1-c_0)^2-(1+c_0)^2}\left(-2c_0\cos kx_2+2i\sin kx_2\right)\phi_0(x_1). \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^l\int_0^\varepsilon |v_\varepsilon(x_1,x_2)|^2dx_1dx_2 \\ &= \frac{2\left|\langle u^i(\cdot,l),\phi_0\rangle\right|^2}{\left|e^{i2kl}(1+c_0)^2-(1-c_0)^2\right|^2}\int_0^l\left|-2c_0\cos kx_2+2i\sin kx_2\right|^2dx_2. \end{aligned}$$

In light of (3.4), it can be shown by a simple calculation that

$$\int_0^l\left|-2c_0\cos kx_2+2i\sin kx_2\right|^2dx_2\le Ck^2l^3,$$

as long as $\varepsilon|\ln\varepsilon|\le l$. On the other hand, similar to the proof in Theorem 3.1, we have

$$\frac{2\left|\langle u^i(\cdot,l),\phi_0\rangle\right|^2}{\left|e^{i2kl}(1-c_0)^2-(1+c_0)^2\right|^2}\le C\frac{\varepsilon}{k^2l^2}$$

for some constant C, if we notice (3.6) and the inequality $|\langle u^i(\cdot,l),\phi_0\rangle|\le\sqrt{\varepsilon}$. Consequently, there exists a constant C_2 such that

$$||v_\varepsilon||^2_{L^2(S_\varepsilon)}\le C_2^2\varepsilon l.$$

Following the same lines above and using (3.4)–(3.6), it can also be shown that

$$||v_\varepsilon||^2_{L^2(S_\varepsilon)}\ge C_1^2\varepsilon l$$

for some constant C_1. We omit the calculations for clarity of exposition. ∎

3.2. Accuracy of the approximation model. To investigate the accuracy of the approximate solution, we first reformulate the models (2.13) and (2.16) in a bounded domain. Let ∂B_R^+ be the half circle with radius R centered at $(\varepsilon/2,l)$, and let ∂B_R^- be the half circle centered at $(\varepsilon/2,0)$ as shown in Figure 3. Here R is a sufficiently large real number independent of ε, l, and λ. Let $\Gamma_R^+=\{(x_1,x_2)\mid |x_1-\varepsilon/2|<R, x_2=l\}$ and $\Gamma_R^-=\{(x_1,x_2)\mid |x_1-\varepsilon/2|<R, x_2=0\}$ be the line segments on the metal planes with length $2R$. We denote the domain bounded by the curves ∂B_R^+ and Γ_R^+ as Ω_R^+ and the domain bounded by ∂B_R^- and Γ_R^- as Ω_R^-, respectively (see Figure 3).

In the exterior domain $\Omega^+\backslash\Omega_R^+$ above the metal plane, by noting the PEC boundary condition, the scattered field can be expressed as the sum of the series

$$u_\varepsilon^s(r^+,\theta)=\sum_{n=0}^\infty\frac{H_n^{(1)}(kr^+)}{H_n^{(1)}(kR)}u^s_{\varepsilon,n}\cos(n\theta),\quad 0<\theta<\pi,$$

where $u^s_{\varepsilon,n}=\frac{2}{\pi}\int_0^\pi u_\varepsilon^s(R,\theta)\cos(n\theta)d\theta$, and $H_n^{(1)}$ is the first kind Hankel function of order n [1]. In the above expansion, the polar coordinate (r^+,θ) is adopted such that

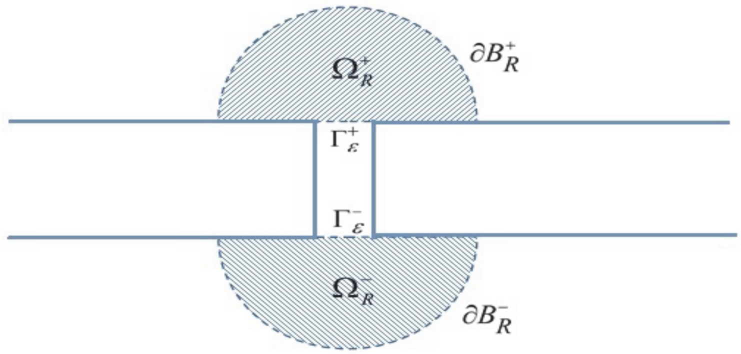

FIG. 3. *Geometry in the bounded domain. The bounded domain* $\Omega_R := \Omega_R^+ \cup \Omega_R^-$.

$r^+ = \sqrt{(x_1 - \varepsilon/2)^2 + (x_2 - l)^2}$. Then the normal derivative on ∂B_R^+ can be written as

$$\frac{\partial u_\varepsilon^s}{\partial r^+}(R,\theta) = \sum_{n=0}^{\infty} \frac{k(H_n^{(1)})'(kR)}{H_n^{(1)}(kR)} u_{\varepsilon,n}^s \cos(n\theta).$$

For $\psi \in H^{\frac{1}{2}}(\partial B_R^\pm)$, we thus define the Dirichlet-to-Neumann map on ∂B_R^+ as

$$(\Lambda^+\psi)(\theta) = \sum_{n=0}^{\infty} \frac{k(H_n^{(1)})'(kR)}{H_n^{(1)}(kR)} \psi_n \cos(n\theta), \quad 0 < \theta < \pi, \tag{3.8}$$

wherein $\psi_n = \frac{2}{\pi}\int_0^\pi \psi(\theta)\cos(n\theta)d\theta$. It follows that the normal derivative of the total field on ∂B_R^+ can be recast as

$$\frac{\partial u_\varepsilon}{\partial n} = \frac{\partial u_\varepsilon^s}{\partial n} + \frac{\partial u^i}{\partial n} + \frac{\partial u^r}{\partial n} = \Lambda^+(u_\varepsilon) + g^+, \tag{3.9}$$

where $g^+ = -\Lambda^+(u^i + u^r) + \frac{\partial u^i}{\partial n} + \frac{\partial u^r}{\partial n}$. Similarly, by using the polar coordinate (r^-, θ) with $r^- = \sqrt{(x_1 - \varepsilon/2)^2 + x_2^2}$, the Dirichlet-to-Neumann map on ∂B_R^- can be expressed as

$$(\Lambda^-\psi)(\theta) = \sum_{n=0}^{\infty} \frac{k(H_n^{(1)})'(kR)}{H_n^{(1)}(kR)} \psi_n \cos(n\theta), \quad \pi < \theta < 2\pi, \tag{3.10}$$

where $\psi_n = \frac{2}{\pi}\int_\pi^{2\pi} \psi(\theta)\cos(n\theta)d\theta$. It follows that

$$\frac{\partial u_\varepsilon}{\partial n} = \frac{\partial u^s}{\partial r^-} = \Lambda^-(u_\varepsilon) \quad \text{on} \quad \partial B_R^-. \tag{3.11}$$

LEMMA 3.3. *The Dirichlet-to-Neumann maps* Λ^+ *and* Λ^- *defined in* (3.8) *and* (3.10) *are bounded from* $H^{\frac{1}{2}}(\partial B_R^+)$ *to* $H^{-\frac{1}{2}}(\partial B_R^+)$ *and from* $H^{\frac{1}{2}}(\partial B_R^-)$ *to* $H^{-\frac{1}{2}}(\partial B_R^-)$, *respectively.*

The readers are referred to [3] for the proof of the lemma. Combining (3.9) and (3.11), we may reformulate the mathematical model (2.13) in the bounded domain

$\Omega_R := \Omega_R^+ \cup \Omega_R^-$ as follows:

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \qquad \text{in } \Omega_R, \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 \qquad \text{on } (\Gamma_R^+ \cup \Gamma_R^-) \backslash (\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-), \\ \dfrac{\partial u_\varepsilon}{\partial n} = \Lambda_\varepsilon^\pm(\mathbf{u}_\varepsilon) \quad \text{on } \Gamma_\varepsilon^\pm, \quad \dfrac{\partial u_\varepsilon}{\partial n} = \Lambda^\pm(u_\varepsilon) + g^\pm \quad \text{on } \partial B_R^\pm. \end{cases} \tag{3.12}$$

Here $g^- = 0$ on ∂B_R^- by noting that the incident field comes only from above. Accordingly, with the Direichlet-to-Neumann maps on $\partial B_R^\pm$, the approximate model (2.16) in the domain Ω_R can be recast as

$$\begin{cases} \Delta v_\varepsilon + k^2 v_\varepsilon = 0 \qquad \text{in } \Omega_R, \\ \dfrac{\partial v_\varepsilon}{\partial n} = 0 \qquad \text{on } (\Gamma_R^+ \cup \Gamma_R^-) \backslash (\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-), \\ \dfrac{\partial v_\varepsilon}{\partial n} = \tilde{\Lambda}_\varepsilon^\pm(\mathbf{v}_\varepsilon) \quad \text{on } \Gamma_\varepsilon^\pm, \quad \dfrac{\partial v_\varepsilon}{\partial n} = \Lambda^\pm(v_\varepsilon) + g^\pm \quad \text{on } \partial B_R^\pm. \end{cases} \tag{3.13}$$

The well-posedness of the problems (2.13) and (3.13) can be argued in a standard way by the use of Gårding type inequalities and the Fredholm alternative [10]. Finally, the wave fields inside the slit S_ε for u_ε and v_ε are given by expansions (2.6) and (2.17)–(2.18), respectively, wherein the Fourier coefficients can be calculated from the solutions obtained in (3.12) and (3.13), respectively. The following theorem states the accuracy of the approximated wave fields in the slit by using the model (3.13) and (2.17)–(2.18).

THEOREM 3.4. *Let u_ε be the solution of* (3.12) *and* (2.6)*, and let v_ε be the solution of the approximate model* (3.13) *and* (2.17)*–*(2.18)*. If ε is sufficiently small, then inside the slit,*

$$||\nabla u_\varepsilon - \nabla v_\varepsilon||_{L^2(S_\varepsilon)} \le C(\lambda, l, R)\, \varepsilon$$

and

$$||u_\varepsilon - v_\varepsilon||_{L^2(S_\varepsilon)} \le C(\lambda, l, R)\, \varepsilon \sqrt{\varepsilon |\ln \varepsilon|}$$

for some positive constant C independent of ε.

We postpone the proof of Theorem 3.4 to section 4.

Remark. Theorem 3.4 also indicates a numerical method to calculate the wave field u_ε in the silt when the gap apertures are small. This can be accomplished by solving the approximate model (3.13) and using the formulas (2.17)–(2.18).

3.3. Estimation of the electromagnetic field enhancement. We give the proof of Theorems 1.1 and 1.2 on the estimation of the electromagnetic field enhancement. It is observed that if $\varepsilon \ll l \ll \lambda$, a combination of Theorems 3.1 and 3.4 yields

$$\begin{aligned} ||\nabla u_\varepsilon||_{L^2(S_\varepsilon)} &\le ||\nabla v_\varepsilon||_{L^2(S_\varepsilon)} + ||\nabla u_\varepsilon - \nabla v_\varepsilon||_{L^2(S_\varepsilon)} \\ &\le C_2 \sqrt{\frac{\varepsilon}{l}} + C(\lambda, l, R)\varepsilon. \end{aligned}$$

Hence

$$||\nabla u_\varepsilon||_{L^2(S_\varepsilon)} \le \frac{3C_2}{2}\sqrt{\frac{\varepsilon}{l}},$$

provided that ε is sufficiently small such that $\sqrt{\varepsilon l}\; C(\lambda, l, R) \le \frac{C_2}{2}$. On the other hand, if $\sqrt{\varepsilon l}\; C(\lambda, l, R) \le \frac{C_1}{2}$, then

$$||\nabla u_\varepsilon||_{L^2(S_\varepsilon)} \ge ||\nabla v_\varepsilon||_{L^2(S_\varepsilon)} - ||\nabla u_\varepsilon - \nabla v_\varepsilon||_{L^2(S_\varepsilon)} \ge \frac{C_1}{2}\sqrt{\frac{\varepsilon}{l}}.$$

In the absence of the metal films, the magnetic field is simply the incident wave u^i inside the slit. A straightforward calculation give rises to

$$||\nabla u^i||_{L^2(S_\varepsilon)} = k\sqrt{\varepsilon l}.$$

Therefore, by using the incident field as the reference field, it is obtained that

$$\frac{C_1}{4\pi}\frac{\lambda}{l} \le \frac{||\nabla u_\varepsilon||_{L^2(S_\varepsilon)}}{||\nabla u^i||_{L^2(S_\varepsilon)}} \le \frac{3C_2}{4\pi}\frac{\lambda}{l}.$$

Now the estimate for the associated electric field enhancement follows by noting that $|\nabla \times H_\varepsilon| = |\nabla u_\varepsilon|$ and the Ampere's law $\nabla \times H_\varepsilon = -i\omega\tau_0 E_\varepsilon$. We arrive at

$$\frac{C_1}{4\pi}\frac{\lambda}{l} \le \frac{||E_\varepsilon||_{L^2(S_\varepsilon)}}{||E^i||_{L^2(S_\varepsilon)}} \le \frac{3C_2}{4\pi}\frac{\lambda}{l}.$$

For the magnetic field, by Theorems 3.2 and 3.4, it follows that

$$\frac{C_1}{2}\sqrt{\varepsilon l} \le ||u_\varepsilon||_{L^2(S_\varepsilon)} \le \frac{3C_2}{2}\sqrt{\varepsilon l},$$

as long as $\frac{C(\lambda,l,R)}{\sqrt{l}}\;\varepsilon\sqrt{|\ln\varepsilon|} \le \min\left\{\frac{C_1}{2}, \frac{C_2}{2}\right\}$. It is clear that $||u^i||_{L^2(S_\varepsilon)} = \sqrt{\varepsilon l}$. Therefore,

$$\frac{C_1}{2} \le \frac{||u_\varepsilon||_{L^2(S_\varepsilon)}}{||u^i||_{L^2(S_\varepsilon)}} \le \frac{3C_2}{2},$$

and the estimate for the magnetic field enhancement follows.

4. Proof of Theorem 3.4.

We begin with some notation that will be used throughout this section. Let $(\cdot,\cdot)_{\Omega_R}$ denote the inner product on $L^2(\Omega_R)$. For a real number s, $H^s(\Omega_R)$ stands for the standard Sobolev space defined on Ω_R equipped with suitable norms [2]. $C(\lambda, l, R)$ denotes a generic constant depending on λ, l, and R only. Its value may vary from step to step but should be clear in the context. To prove Theorem 3.4, we extend the arguments in [12] for the case of the semi-infinite slit to the finite-thickness slit considered in this paper.

4.1. Estimate of $\|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega_R)}$.

Let u_0 be the solution of (3.12) when there is no slit, namely, $\varepsilon = 0$. Then u_0 solves the variational problem

$$a_0(u_0, w) = b(w) \quad \forall w \in H^1(\Omega_R). \tag{4.1}$$

The bilinear form

$$a_0(u_0, w) = (\nabla u_0, \nabla w)_{\Omega_R} - k^2(u_0, w)_{\Omega_R} - \langle \Lambda^+ u_0, w\rangle_{\partial B_R^+} - \langle \Lambda^- u_0, w\rangle_{\partial B_R^-},$$

and the linear functional $b(w) = \langle g, w\rangle_{\partial B_R^+}$. Similarly, u_ε is the solution of the variational problem

$$a_\varepsilon(u_\varepsilon, w) = b(w) \quad \forall w \in H^1(\Omega_R), \tag{4.2}$$

where the bilinear form

$$\begin{aligned} a_\varepsilon(u_\varepsilon, w) = (\nabla u_\varepsilon, \nabla w)_{\Omega_R} - k^2(u_\varepsilon, w)_{\Omega_R} + \langle \Lambda_\varepsilon^+ \mathbf{u}_\varepsilon, w^+\rangle_{\Gamma_\varepsilon^+} + \langle \Lambda_\varepsilon^- \mathbf{u}_\varepsilon, w^-\rangle_{\Gamma_\varepsilon^-} \\ -\langle \Lambda^+ u_\varepsilon, w\rangle_{\partial B_R^+} - \langle \Lambda^- u_\varepsilon, w\rangle_{\partial B_R^-}. \end{aligned} \tag{4.3}$$

By a direct comparison, it follows that

$$a_\varepsilon(u_\varepsilon - u_0, w) = a_0(u_0, w) - a_\varepsilon(u_0, w).$$

From the boundness of the Dirichelt-to-Neumann operators $\tilde{\Lambda}_\varepsilon^+$, $\tilde{\Lambda}_\varepsilon^-$, Λ_ε^+, and Λ_ε^-, we have

$$\begin{aligned} &|a_0(u_0, w) - a_\varepsilon(u_0, w)| \\ &= |\langle \Lambda_\varepsilon^+ \mathbf{u}_0, w^+\rangle_{\Gamma_\varepsilon^+} + \langle \Lambda_\varepsilon^- \mathbf{u}_0, w^-\rangle_{\Gamma_\varepsilon^-}| \\ &\le |\langle \tilde{\Lambda}_\varepsilon^+ \mathbf{u}_0, w^+\rangle_{\Gamma_\varepsilon^+} + \langle \tilde{\Lambda}_\varepsilon^- \mathbf{u}_0, w^-\rangle_{\Gamma_\varepsilon^-}| \\ &\quad + |\langle (\Lambda_\varepsilon^+ - \tilde{\Lambda}_\varepsilon^+)\mathbf{u}_0, w^+\rangle_{\Gamma_\varepsilon^+} + \langle (\Lambda_\varepsilon^- - \tilde{\Lambda}_\varepsilon^-)\mathbf{u}_0, w^-\rangle_{\Gamma_\varepsilon^-}| \\ &\le \frac{k}{|\sin(kl)|}(|u_{0,0}^+| + |u_{0,0}^-|)(|w_{0,0}^+| + |w_{0,0}^-|) + C(\lambda, l, R)\, |\mathbf{u}_0|_{H^{\frac12}(\Gamma_\varepsilon)} |\mathbf{w}|_{H^{\frac12}(\Gamma_\varepsilon)} \\ &\le C(\lambda, l, R)\left(\varepsilon\sqrt{|\ln\varepsilon|}\, \|u_0\|_{H^2(\Omega_R)} \|w\|_{H^1(\Omega_R)} + \varepsilon\, \|u_0\|_{H^3(\Omega_R)} \|w\|_{H^1(\Omega_R)}\right), \end{aligned} \tag{4.4}$$

where the last inequality is obtained by applying Lemmas A.1 and A.2. Note that $u_0 = u^i + u^r$ in the absence of a slit. Hence, it is straightforward that

$$|a_0(u_0, w) - a_\varepsilon(u_0, w)| \le C(\lambda, l, R)\, \varepsilon\sqrt{|\ln\varepsilon|}\, \|w\|_{H^1(\Omega_R)}.$$

On the other hand, let L_ε be the induced operator for the bilinear form (4.3) such that $a_\varepsilon(u_\varepsilon, w) = (L_\varepsilon u_\varepsilon, w)$. From the boundness of the inverse of L_ε (Lemma A.3), it is obtained that there exists a positive constant ε_0 such that for $0 < \varepsilon < \varepsilon_0$,

$$\|u_\varepsilon - u_0\|_{H^1(\Omega_R)} \le C(\lambda, l, R)\, \varepsilon\sqrt{|\ln\varepsilon|}. \tag{4.5}$$

It is observed that the weak solution of (3.13) satisfies

$$\tilde{a}_\varepsilon(v_\varepsilon, w) = b(w) \quad \forall w \in H^1(\Omega_R), \tag{4.6}$$

where the bilinear form

$$\begin{aligned} \tilde{a}_\varepsilon(u_\varepsilon, w) = (\nabla u_\varepsilon, \nabla w)_{\Omega_R} - k^2(u_\varepsilon, w)_{\Omega_R} + \langle \tilde{\Lambda}_\varepsilon^+ \mathbf{u}_\varepsilon, w^+\rangle_{\Gamma_\varepsilon^+} + \langle \tilde{\Lambda}_\varepsilon^- \mathbf{u}_\varepsilon, w^-\rangle_{\Gamma_\varepsilon^-} \\ - \langle \Lambda^+ u_\varepsilon, w\rangle_{\partial B_R^+} - \langle \Lambda^- u_\varepsilon, w\rangle_{\partial B_R^-}. \end{aligned} \tag{4.7}$$

Following the same lines as above, it can be shown that for sufficiently small ε,

$$\|v_\varepsilon - u_0\|_{H^1(\Omega_R)} \le C(\lambda, l, R)\, \varepsilon\sqrt{|\ln\varepsilon|}. \tag{4.8}$$

In what follows, we derive the estimation for $\|v_\varepsilon - u_0\|_{H^{\frac{3}{2}}(\Omega_R)}$. Letting $\xi = v_\varepsilon - u_0$, we see that ξ is the solution of the following boundary value problem:

$$\begin{cases} \Delta\xi + k^2\xi = 0 & \text{in } \Omega_R^+ \cup \Omega_R^-, \\ \dfrac{\partial\xi}{\partial n} = 0 & \text{on } (\Gamma_R^+ \cup \Gamma_R^-)\backslash(\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-), \\ \dfrac{\partial\xi}{\partial n} = \tilde{\Lambda}_\varepsilon^\pm(\mathbf{v}_\varepsilon) \quad \text{on } \Gamma_\varepsilon^\pm, & \dfrac{\partial\xi}{\partial n} = \Lambda^\pm(\xi) \quad \text{on } \partial B_R^\pm. \end{cases}$$

The standard elliptic regularity estimate for the Neumann problem leads to

$$\begin{aligned} \|\xi\|_{H^{\frac{3}{2}}(\Omega_R)} &\le C(\lambda,l,R)\left\|\left[\,\tilde{\Lambda}_\varepsilon^+(\mathbf{v}_\varepsilon), \tilde{\Lambda}_\varepsilon^-(\mathbf{v}_\varepsilon)\,\right]\right\|_{L^2(\Gamma_\varepsilon)} \\ &\le C(\lambda,l,R)\left((|u_{0,0}^+| + |u_{0,0}^-|) + (|v_{\varepsilon,0}^+ - u_{0,0}^+| + |v_{\varepsilon,0}^- - u_{0,0}^-|)\right)\|\phi_0\|_{L^2(0,\varepsilon)} \\ &\le C(\lambda,l,R)\left(\sqrt{\varepsilon}\,\|u_0\|_{H^2(\Omega_R)} + \sqrt{\varepsilon|\ln\varepsilon|}\,\|v_\varepsilon - u_0\|_{H^1(\Omega_R)}\right), \end{aligned}$$

where we have used Lemma A.2. Consequently, by virtue of (4.8), it is obtained that

$$\|v_\varepsilon - u_0\|_{H^{\frac{3}{2}}(\Omega_R)} \le C(\lambda,l,R)\,\sqrt{\varepsilon}. \tag{4.9}$$

Now we are ready to estimate $\|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega_R)}$. From (4.2) and (4.6), it is observed that

$$a_\varepsilon(u_\varepsilon - v_\varepsilon, w) = \tilde{a}_\varepsilon(v_\varepsilon, w) - a_\varepsilon(v_\varepsilon, w). \tag{4.10}$$

In addition, a direct comparison of (4.3) and (4.7) yields

$$\begin{aligned} |\tilde{a}_\varepsilon(v_\varepsilon,w) - a_\varepsilon(v_\varepsilon,w)| &\le |\langle(\Lambda_\varepsilon^+ - \tilde{\Lambda}_\varepsilon^+)\mathbf{v}_\varepsilon, w^+\rangle_{\Gamma_\varepsilon^+}| + |\langle(\Lambda_\varepsilon^- - \tilde{\Lambda}_\varepsilon^-)\mathbf{v}_\varepsilon, w^-\rangle_{\Gamma_\varepsilon^-}| \\ &\le |\langle(\Lambda_\varepsilon^+ - \tilde{\Lambda}_\varepsilon^+)\mathbf{u}_0, w^+\rangle_{\Gamma_\varepsilon^+}| + |\langle(\Lambda_\varepsilon^- - \tilde{\Lambda}_\varepsilon^-)\mathbf{u}_0, w^-\rangle_{\Gamma_\varepsilon^-}| \\ &\quad + |\langle(\Lambda_\varepsilon^+ - \tilde{\Lambda}_\varepsilon^+)(\mathbf{v}_\varepsilon - \mathbf{u}_0), w^+\rangle_{\Gamma_\varepsilon^+}| \\ &\quad + |\langle(\Lambda_\varepsilon^- - \tilde{\Lambda}_\varepsilon^-)(\mathbf{v}_\varepsilon - \mathbf{u}_0), w^-\rangle_{\Gamma_\varepsilon^-}|. \end{aligned}$$

In light of the estimates in (4.4), it follows that

$$|\langle(\Lambda_\varepsilon^+ - \tilde{\Lambda}_\varepsilon^+)\mathbf{u}_0, w^+\rangle_{\Gamma_\varepsilon^+}| + |\langle(\Lambda_\varepsilon^- - \tilde{\Lambda}_\varepsilon^-)\mathbf{u}_0, w^-\rangle_{\Gamma_\varepsilon^-}| \le C(\lambda,l,R)\,\varepsilon\;\|w\|_{H^1(\Omega_R)}.$$

On the other hand, from the boundness of the Dirichelt-to-Neumann operators $\tilde{\Lambda}_\varepsilon^+$, $\tilde{\Lambda}_\varepsilon^-$, Λ_ε^+, and Λ_ε^-,

$$\begin{aligned} &|\langle(\Lambda_\varepsilon^+ - \tilde{\Lambda}_\varepsilon^+)(\mathbf{v}_\varepsilon - \mathbf{u}_0), w^+\rangle_{\Gamma_\varepsilon^+}| + |\langle(\Lambda_\varepsilon^- - \tilde{\Lambda}_\varepsilon^-)(\mathbf{v}_\varepsilon - \mathbf{u}_0), w^-\rangle_{\Gamma_\varepsilon^-}| \\ &\le C(\lambda,l,R)\,|\mathbf{v}_\varepsilon - \mathbf{u}_0|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}\,|\mathbf{w}|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \\ &\le C(\lambda,l,R)\sqrt{\varepsilon}\,\|v_\varepsilon - u_0\|_{H^{\frac{3}{2}}(\Omega_R)}\,\|w\|_{H^1(\Omega_R)} \\ &\le C(\lambda,l,R)\,\varepsilon\;\|w\|_{H^1(\Omega_R)}, \end{aligned}$$

where we have used Lemma A.1 and (4.9). For the variational problem (4.10), a combination of the above inequalities and the boundness of the inverse of the linear operator L_ε induced by the bilinear form $a(u_\varepsilon, w)$ (Lemma A.3) leads to the following desired estimate:

$$\|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega_R)} \le C(\lambda,l,R)\,\varepsilon. \tag{4.11}$$

4.2. Estimate of $||\nabla u_\varepsilon - \nabla v_\varepsilon||_{L^2(S_\varepsilon)}$ and $||u_\varepsilon - v_\varepsilon||_{L^2(S_\varepsilon)}$. Note that u_ε and v_ε in the slit S_ε are given by (2.1) and (2.17), respectively. From the Parseval's identity,

$$\begin{aligned}
&||\nabla u_\varepsilon - \nabla v_\varepsilon||^2_{L^2(S_\varepsilon)} \\
&= \int_0^l k^2 \left|(\alpha_0^- - \tilde{\alpha}_0^-)e^{ikx_2} - (\alpha_0^+ - \tilde{\alpha}_0^+)e^{-ik(x_2-l)}\right|^2 dx_2 \\
&\quad + \int_0^l \sum_{n=1}^{\infty} \left(\frac{n\pi}{\varepsilon}\right)^2 \left|\alpha_n^- e^{i\gamma_n x_2} + \alpha_n^+ e^{-i\gamma_n(x_2-l)}\right|^2 \\
&\quad + (i\gamma_n)^2 \left|\alpha_n^- e^{i\gamma_n x_2} - \alpha_n^+ e^{-i\gamma_n(x_2-l)}\right|^2 dx_2 \\
&\le 2k^2 l\left(|\alpha_0^- - \tilde{\alpha}_0^-|^2 + |\alpha_0^+ - \tilde{\alpha}_0^+|^2\right) + \sum_{n=1}^{\infty} \frac{1+2\left(\frac{n\pi}{\varepsilon}\right)^2}{\frac{n\pi}{2\varepsilon}} (|\alpha_n^-|^2 + |\alpha_n^+|^2).
\end{aligned}$$

For $n = 0$, by using the fact that $kl \ll 1$, it is obtained that

$$\begin{aligned}
|\alpha_0^- - \tilde{\alpha}_0^-|^2 &= \frac{\left|e^{ikl}(u_{\varepsilon,0}^+ - v_{\varepsilon,0}^+) - (u_{\varepsilon,0}^- - v_{\varepsilon,0}^-)\right|^2}{\left|e^{i2kl} - 1\right|^2} \\
&\le \frac{4}{k^2 l^2}\left(|u_{\varepsilon,0}^+ - v_{\varepsilon,0}^+|^2 + |u_{\varepsilon,0}^- - v_{\varepsilon,0}^-|^2\right) \qquad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
|\alpha_0^+ - \tilde{\alpha}_0^+|^2 &= \frac{\left|e^{ikl}(u_{\varepsilon,0}^- - v_{\varepsilon,0}^-) - (u_{\varepsilon,0}^+ - v_{\varepsilon,0}^+)\right|^2}{\left|e^{i2kl} - 1\right|^2} \\
&\le \frac{4}{k^2 l^2}\left(|u_{\varepsilon,0}^+ - v_{\varepsilon,0}^+|^2 + |u_{\varepsilon,0}^- - v_{\varepsilon,0}^-|^2\right). \qquad (4.13)
\end{aligned}$$

From (2.5), and $i\gamma_n = -[(n\pi/\epsilon)^2 - k^2]^{1/2}$, it is observed that for $n \ge 1$,

$$|\alpha_n^-|^2 = \frac{\left|e^{i\gamma_n l} u_{\varepsilon,n}^+ - u_{\varepsilon,n}^-\right|^2}{\left|e^{i2\gamma_n l} - 1\right|^2} \le 4(|u_{\varepsilon,n}^+|^2 + |u_{\varepsilon,n}^-|^2). \qquad (4.14)$$

Similarly,

$$|\alpha_n^+|^2 \le 4(|u_{\varepsilon,n}^+|^2 + |u_{\varepsilon,n}^-|^2). \qquad (4.15)$$

Consequently,

$$\begin{aligned}
&||\nabla u_\varepsilon - \nabla v_\varepsilon||^2_{L^2(S_\varepsilon)} \\
&\le C(\lambda, l, R)\left\{|u_{\varepsilon,0}^+ - v_{\varepsilon,0}^+|^2 + |u_{\varepsilon,0}^- - v_{\varepsilon,0}^-|^2 + \sum_{n=1}^{\infty}\left(1 + \left(\frac{n\pi}{\varepsilon}\right)^2\right)^{1/2} \left(|u_{\varepsilon,n}^+|^2 + |u_{\varepsilon,n}^-|^2\right)\right\} \\
&= C(\lambda, l, R)\left\{|u_{\varepsilon,0}^+ - v_{\varepsilon,0}^+|^2 + |u_{\varepsilon,0}^- - v_{\varepsilon,0}^-|^2 + |\mathbf{u}_\varepsilon|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}\right\}.
\end{aligned}$$

From Lemma A.1, (4.9), and (4.11), we note that

$$\begin{aligned}
|\mathbf{u}_\varepsilon|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} &\le |\mathbf{u}_0|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} + |\mathbf{v}_\varepsilon - \mathbf{u}_0|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} + |\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \\
&\le C(\lambda, l, R)\left(\varepsilon \|u_0\|_{H^3(\Omega_R)} + \sqrt{\varepsilon}\, \|v_\varepsilon - u_0\|_{H^{\frac{3}{2}}(\Omega_R)} + \|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega_R)}\right) \\
&\le C(\lambda, l, R)\, \varepsilon. \qquad (4.16)
\end{aligned}$$

On the other hand, by Lemma A.2,

$$
\begin{aligned}
|u^+_{\varepsilon,0} - v^+_{\varepsilon,0}| + |u^-_{\varepsilon,0} - v^-_{\varepsilon,0}| &\le C(\lambda, l, R)\sqrt{\varepsilon|\ln\varepsilon|}\, \|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega_R)} \\
&\le C(\lambda, l, R)\, \varepsilon\sqrt{\varepsilon|\ln\varepsilon|}.
\end{aligned} \tag{4.17}
$$

Therefore,

$$||\nabla u_\varepsilon - \nabla v_\varepsilon||_{L^2(S_\varepsilon)} \le C(\lambda, l, R)\, \varepsilon.$$

The estimate for $||u_\varepsilon - v_\varepsilon||_{L^2(S_\varepsilon)}$ follows a similar fashion. An application of the Parseval's identity and (4.14)–(4.13) yields

$$
\begin{aligned}
||u_\varepsilon - v_\varepsilon||^2_{L^2(S_\varepsilon)} &\le 2l\left(|\alpha_0^- - \tilde{\alpha}_0^-|^2 + |\alpha_0^+ - \tilde{\alpha}_0^+|^2\right) + \sum_{n=1}^{\infty} \frac{2\varepsilon}{n\pi}(|\alpha_n^-|^2 + |\alpha_n^+|^2) \\
&\le C(\lambda, l, R)\left\{ |u^+_{\varepsilon,0} - v^+_{\varepsilon,0}|^2 + |u^-_{\varepsilon,0} - v^-_{\varepsilon,0}|^2 \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} \frac{\varepsilon}{n\pi}\left(|u^+_{\varepsilon,n}|^2 + |u^-_{\varepsilon,n}|^2\right) \right\}.
\end{aligned}
$$

Furthermore, we see that

$$
\begin{aligned}
&||u_\varepsilon - v_\varepsilon||^2_{L^2(S_\varepsilon)} \\
&\le C(\lambda, l, R)\left\{ |u^+_{\varepsilon,0} - v^+_{\varepsilon,0}|^2 + |u^-_{\varepsilon,0} - v^-_{\varepsilon,0}|^2 \right. \\
&\qquad \left. + \varepsilon^2 \sum_{n=1}^{\infty} \left(1 + \left(\frac{n\pi}{\varepsilon}\right)^2\right)^{1/2} \left(|u^+_{\varepsilon,n}|^2 + |u^-_{\varepsilon,n}|^2\right) \right\} \\
&= C(\lambda, l, R)\left\{ |u^+_{\varepsilon,0} - v^+_{\varepsilon,0}|^2 + |u^-_{\varepsilon,0} - v^-_{\varepsilon,0}|^2 + \varepsilon^2\, |\mathbf{u}_\varepsilon|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \right\}.
\end{aligned}
$$

Therefore, by virtue of (4.16) and (4.17), the following inequality holds:

$$||u_\varepsilon - v_\varepsilon||_{L^2(S_\varepsilon)} \le 2C(\lambda, l, R)\, \varepsilon\sqrt{\varepsilon|\ln\varepsilon|}.$$

5. Conclusion. We have investigated the electromagnetic field enhancement in the classical regime when an electromagnetic wave impinges upon a nanogap. It is shown that the electric field enhancement occurs inside the slit with an order of $O(\lambda/l)$ when the gap size is sufficiently small. Thus the enhancement is enormous when the metal film is thin such that $l \ll \lambda$. In addition, such an enhancement strength remains true when the gap size approaches zero. On the other hand, there is no significant magnetic field enhancement in the nano slit. One direction for this research is the study of the electromagnetic field enhancement for the three-dimensional nanogaps. This is our ongoing work and will be reported elsewhere.

Appendix.

LEMMA A.1. *Let* $\mathbf{u} = [u(x_1, l), u(x_1, 0)]$, *and let the semi-norm* $|\mathbf{u}|_{H^s(\Gamma_\varepsilon)}$ *be defined as* (2.4).

(1) *If* $u \in H^1(\Omega_R)$, *then* $|\mathbf{u}|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \le C\,\|u\|_{H^1(\Omega_R)}$.

(2) *If* $u \in H^{\frac{3}{2}}(\Omega_R)$, *then* $|\mathbf{u}|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \le C\sqrt{\varepsilon}\,\|u\|_{H^{\frac{3}{2}}(\Omega_R)}$.

(3) *If* $u \in H^{2+\delta}(\Omega_R)$ *for* $\delta > 0$, *then* $|\mathbf{u}|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \le C\varepsilon\,\|u\|_{H^{2+\delta}(\Omega_R)}$.

Proof. Item (1) follows directly from the trace theorem. To show (2), we set

$$\bar{\mathbf{u}} = [u(x_1, l) - u^+_{\varepsilon,0}\ \phi_0(x_1),\ u(x_1, 0) - u^-_{\varepsilon,0}\ \phi_0(x_1)].$$

From the Bramble–Hilbert lemma [2], it follows that

$$\|\bar{\mathbf{u}}\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon \|\mathbf{u}'\|_{L^2(\Gamma_\varepsilon)} \quad \text{and} \quad \|\bar{\mathbf{u}}\|_{H^1(\Gamma_\varepsilon)} \le C \|\mathbf{u}'\|_{L^2(\Gamma_\varepsilon)},$$

where $\mathbf{u}' = [u'(x_1, l), u'(x_1, 0)]$. An application of the interpolation inequality and the trace theorem leads to

$$\|\bar{\mathbf{u}}\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \le C\sqrt{\varepsilon}\, \|\mathbf{u}'\|_{L^2(\Gamma_\varepsilon)} \le C\sqrt{\varepsilon}\, \|u\|_{H^{\frac{3}{2}}(\Omega_R)}. \tag{A.1}$$

By the definition of $\bar{\mathbf{u}}$, it is obvious that $|\mathbf{u}|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} = \|\bar{\mathbf{u}}\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}$. Thus the desired inequality follows.

If $u \in H^{2+\delta}(\Omega_R)$, by the Sobolev embedding theorem, we have $\|\nabla u\|_{L^\infty(\overline{\Omega}_R)} \le C \|u\|_{H^{2+\delta}(\Omega_R)}$. Using (A.1), it is obtained that

$$\|\bar{\mathbf{u}}\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \le C\sqrt{\varepsilon}\, \|\mathbf{u}'\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon \|\mathbf{u}'\|_{L^\infty(\Gamma_\varepsilon)} \le C\varepsilon \|u\|_{H^{2+\delta}(\Omega_R)}.$$

This completes the proof. ∎

LEMMA A.2. *Let* $u^+ = u(x_1, l)$, $u^- = u(x_1, 0)$ *and* $u^+_0 = \langle u^+, \phi_0\rangle$, $u^-_0 = \langle u^-, \phi_0\rangle$.

(1) $|u^+_0| + |u^-_0| \le C\sqrt{\varepsilon|\ln\varepsilon|}\, \|u\|_{H^1(\Omega_R)}$ *if* $u \in H^1(\Omega_R)$.

(2) $|u^+_0| + |u^-_0| \le C\sqrt{\varepsilon}\, \|u\|_{H^2(\Omega_R)}$ *if* $u \in H^2(\Omega_R)$.

Proof. We first extend $\phi_0(x_1)$ to $\mathbb{R}$ by defining it as 0 outside the interval $(0, \varepsilon)$. Let $u(x_1, l)$ be the trace of u on Γ^+_R, and extend it to $\mathbb{R}$ suitably such that $\|u(\cdot, l)\|_{H^s(\mathbb{R})} \le C \|u(\cdot, l)\|_{H^s(\Gamma^+_R)}$ for $s = 0$ and $s = \frac{3}{2}$. From the definition of u^+_0 and the trace theorem, it follows that

$$|u^+_0| = |\langle u_\varepsilon(\cdot, l), \phi_0\rangle_{L^2(\mathbb{R})}| \le \|u(\cdot, l)\|_{H^s(\mathbb{R})} \|\phi_0\|_{H^{-s}(\mathbb{R})} \le C \|u\|_{H^{s+\frac{1}{2}}(\Omega_R)} \|\phi_0\|_{H^{-s}(\mathbb{R})}.$$

Note that $\hat{\phi_0}(\xi) = \sqrt{\varepsilon}\ \frac{e^{i\varepsilon\xi}-1}{i\varepsilon\xi}$. Thus

$$\|\phi_0\|^2_{H^{-s}(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{1}{(1+|\xi|^2)^s} \left|\frac{e^{i\varepsilon\xi}-1}{i\varepsilon\xi}\right|^2 d\xi,$$

and a direct calculation yields the desired inequalities for sufficiently small ε. We refer to [12] for more details. The estimate for $|u^-_0|$ follows in a similar fashion. ∎

LEMMA A.3. *Let* L_ε *and* $\tilde{L}_\varepsilon$ *be the induced linear operator for the bilinear form* (4.3) *and* (4.7)*, respectively, such that* $a_\varepsilon(u_\varepsilon, w) = (L_\varepsilon u_\varepsilon, w)$ *and* $\tilde{a}_\varepsilon(u_\varepsilon, w) = (\tilde{L}_\varepsilon u_\varepsilon, w)$. *There exists a positive constant* ε_0 *that for all* $0 < \varepsilon < \varepsilon_0$,

$$\|L^{-1}_\varepsilon\|_{\mathcal{L}(H^1(\Omega_R))} \le C(\lambda, l) \quad \text{and} \quad \left\|\tilde{L}^{-1}_\varepsilon\right\|_{\mathcal{L}(H^1(\Omega_R))} \le C(\lambda, l),$$

where $C(\lambda, l)$ *is some positive constant independent of* ε.

We provide a proof by contradiction following the arguments in [12]. Indeed, if the statement is false, then there is a sequence $\varepsilon_n \to 0$ such that

$$\|u_{\varepsilon_n}\|_{H^1(\Omega_R)} = 1 \quad \text{and}\ \|L_{\varepsilon_n} u_{\varepsilon_n}\|_{H^1(\Omega_R)} \to 0.$$

It is clear that there exists a subsequence, which we still denote as ε_n, such that $u_{\varepsilon_n} \to u_0$ weakly in $H^1(\Omega_R)$ and strongly in $L^2(\Omega_R)$. By taking the limit, it follows that

$$\lim_{\varepsilon_n \to 0} a_{\varepsilon_n}(u_{\varepsilon_n}, w) = a_0(u_0, w) = 0 \quad \forall w \in H^1(\Omega_R).$$

We obtain that $u_0 = 0$. Now an evaluation of $a_\varepsilon(u_{\varepsilon_n}, w) = (L_{\varepsilon_n} u_{\varepsilon_n}, w)$ with $w = u_{\varepsilon_n}$ yields

$$\begin{aligned} &\|\nabla u_{\varepsilon_n}\|^2_{L^2(\Omega_R)} - k^2 \|u_{\varepsilon_n}\|^2_{L^2(\Omega_R)} + \langle \Lambda^+_{\varepsilon_n} \mathbf{u}_{\varepsilon_\mathbf{n}}, u^+_{\varepsilon_n} \rangle_{\Gamma^+_{\varepsilon_n}} + \langle \Lambda^-_{\varepsilon_n} \mathbf{u}_{\varepsilon_\mathbf{n}}, u^-_{\varepsilon_n} \rangle_{\Gamma^-_{\varepsilon_n}} \\ &- \langle \Lambda^+ u_{\varepsilon_n}, u_{\varepsilon_n} \rangle_{\partial B^+_R} - \langle \Lambda^- u_{\varepsilon_n}, u_{\varepsilon_n} \rangle_{\partial B^-_R} = (L_{\varepsilon_n} u_{\varepsilon_n}, u_{\varepsilon_n}). \end{aligned}$$

For sufficiently small ε_n, using the fact that $\mathrm{Re}\langle \Lambda^+ u_{\varepsilon_n}, u_{\varepsilon_n} \rangle_{\partial B^+_R} \leq 0$, $\mathrm{Re}\langle \Lambda^- u_{\varepsilon_n}, u_{\varepsilon_n} \rangle_{\partial B^-_R} \leq 0$, and

$$\lim_{\varepsilon_n \to 0} \langle \Lambda^+_{\varepsilon_n} \mathbf{u}_{\varepsilon_\mathbf{n}}, u^+_{\varepsilon_n} \rangle_{\Gamma^+_{\varepsilon_n}} \geq 0, \quad \lim_{\varepsilon_n \to 0} \langle \Lambda^-_{\varepsilon_n} \mathbf{u}_{\varepsilon_\mathbf{n}}, u^-_{\varepsilon_n} \rangle_{\Gamma^-_{\varepsilon_n}} \geq 0,$$

we have $\|\nabla u_{\varepsilon_n}\|^2_{L^2(\Omega_R)} \to 0$ as $\varepsilon_n \to 0$. This contradicts the assumption that $\|u_{\varepsilon_n}\|_{H^1(\Omega_R)} = 1$.

Acknowledgments. We thank Sang-Hyun Oh and Hyeong-Ryeol Park at the University of Minnesota and Hoai-Minh Nguyen at EPFL for stimulating discussions. We also thank anonymous referees for their valuable suggestions for improving the manuscript.

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