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A multi-frequency inverse source problem

Gang Bao [a,b,*], Junshan Lin [b], Faouzi Triki [c]

[a] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[b] *Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, United States*
[c] *Laboratoire Jean Kuntzmann, Université Joseph Fourier, BP 53, 38041 Grenoble Cedex 9, France*

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ABSTRACT

This paper is concerned with an inverse source problem that determines the source from measurements of the radiated fields away at multiple frequencies. Rigorous stability estimates are established when the background medium is homogeneous. It is shown that the ill-posedness of the inverse problem decreases as the frequency increases. Under some regularity assumptions on the source function, it is further proven that by increasing the frequency, the logarithmic stability converts to a linear one for the inverse source problem.

1. Introduction

The inverse source problem that determines the source from measurements of the radiated field away arises in and motivated by many applications in optoelectronics and biomedical engineering. One example is to use electric and magnetic measurements on the surface of the human head to determine source currents in the brain that produced these measured fields [2,3,17,18,23]. Another example has to do with antenna synthesis. The inverse source problem [4,18,27] in this case is to reconstruct the unknown source (antenna) that is embedded in a known source region B_{R_0} (a given substrate medium), which radiates a measurable exterior field outside B_{R_0}.

Inverse source problems for both the scalar Helmholtz equation and the full vector electromagnetic model have received much attention recently. We refer the reader to [4,18,27] for some recent results and references. The existing results are all concerned with the case where the frequency is fixed. In

* Corresponding author at: Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, United States.

E-mail addresses: bao@math.msu.edu (G. Bao), linjunsh@msu.edu (J. Lin), Faouzi.Triki@imag.fr (F. Triki).

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fact, a majority of the results focused on the static inverse source problem where the frequency is zero. There are two main difficulties associated with the inverse source problem at fixed frequency: non-uniqueness and the ill-posedness. It is well known [19,16,13,2] that there exist an infinite number of sources that radiate fields vanishing identically outside their support volumes. Consequently, for a fixed frequency the volume sources cannot be uniquely determined from surface measurements. Moreover, since the inverse problem is linear, an infinity of solutions can be obtained by adding any one of these non-radiating sources to a given solution. Thus, in order to obtain a unique solution to the inverse source problem, it is necessary to impose additional constraints on the source. A commonly used choice of constraints is to pick up the solution with a minimum L^2 norm (called the minimum energy solution), which corresponds to the L^2-orthogonal projection of the original source onto the space orthogonal to non-radiating sources (which span the nullspace of the inverse source problem). It also represents the pseudo-inverse of the inverse source problem. Since the nullspace only depends on the index of refraction of the background medium where the source is radiating, the difference between the minimum energy solution and the original source also depends on the medium and hence could be significant. This represents a severe disadvantage of the inverse source problem at fixed frequency. The non-uniqueness issue has been examined by several researchers [19,16,13,22,25,30]. Recently, in the vector static case (zero frequency), Hauer, Kühn and Potthast [22] studied the limitations to reconstruct the current density from its magnetic field by characterizing the nullspace of the Biot–Savart operator for anisotropic conductivity. In [2], the non-radiating sources are considered when the background medium is non-constant for the full Maxwell system.

Another main difficulty for the inverse source problem is the ill-posedness, i.e., infinitesimal noise in the measured data may give rise to a large error in the computed minimum energy solution. In practice, regularization algorithms are used to reconstruct the solution with a minimum L^2 norm, for example, the Tikhonov-projection algorithm [25]. Since the singular eigenvalues of the forward linear problem are exponentially decreasing [20], it is expected that these algorithms have a logarithmic convergence. More recently, in [21], Eller and Valdivia considered the inverse problem of identifying the shape and location of a finitely supported source function from measurements of the acoustic field on a closed surface for an infinite unbounded (chosen) frequencies. They showed the uniqueness of the solutions when the set of frequencies coincides with the Dirichlet eigenvalues of the Laplacian. In [20], an observation was made that the L^2 norm of the minimum energy solution in the case of homogeneous media depends critically on the product $c := R_0 k$, where R_0 is the source radius and k is the frequency. The minimum energy solution increases exponentially with decreasing c below a critical limit. This exponential increase of the energy indicates that the ill-posedness in the reconstruction of the original source grows with decreasing c. The result is consistent with physical considerations, especially the uncertainty principle [11].

The goal of the paper is to investigate the multi-frequency inverse source problem. We first address the uniqueness issue and prove that the source can be uniquely determined by observations of the radiated fields outside at a set of frequencies with an accumulation point (not necessarily unbounded). We further establish stability estimates when the measurements are performed for all frequencies less than k_0. Our main result is of two-fold: (1) in the case where the data is collected for all frequencies and $k_0 = 1$, we show that the stability is logarithmic when $c = k_0 R_0$ (R_0 is the source radius) is larger than a critical limit, and is of the Hölder type when c becomes small. Hence for a limited band of frequencies one can improve the stability of the inverse problem by decreasing the source radius R_0. (2) If R_0 is fixed, then a logarithmic stability may be established when c is small. However, in this case, for large values of c, a linear stability result holds. These results are in good agreement with the underlying physics, particularly the uncertainty principle.

Recently, for the related but more challenging nonlinear inverse medium scattering problems, it has been proposed that the ill-posedness of the inverse problem could be overcome by employing multiple frequency data [14,11]. The idea is illustrated by solving linear equations at lowest frequency to obtain low-frequency modes of the medium (Born approximation). Updates are made by using the data at higher frequency sequentially, through linearizations, until a sufficiently high frequency where the dominant modes of the medium are essentially recovered. Computational methods have been developed for solving the inverse medium problems for Helmholtz and Maxwell equations at multiple and fixed frequencies [5–10]. We also refer to [26] for interesting stability analysis for the Helmholtz equation with respect to the wave number.

In [11], under some reasonable assumptions, the authors have established the convergence of the algorithm. We think the results here shed some light on the stability analysis of the inverse medium scattering problem. In fact, the multiple scattering in the inverse medium problem may be viewed as a collection of inverse source problems with increasing frequencies (the product between the incident wave frequency and the index of the medium at certain points).

The outline of the paper is as follows. The formulation of the inverse source problem as well as its ill-posedness analysis are presented in Section 2. Our main results are established in Section 3. We first prove a uniqueness result for the multi-frequency inverse problem. Then, we derive different stability estimates depending on the value of the product $c = k_0 R_0$. The first stability result treats the case where the frequency k_0 is fixed but the source radius R_0 is variable. The second result is concerned with the case with fixed R_0 but variable frequency k_0. The main results are stated in Theorems 3.2 and 3.3, and Corollaries 3.1 and 3.2. In Appendix A, we provide some known results in analysis used in the previous sections.

We complete the introduction by introducing some general notations and definitions that are used throughout the paper. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. Here, our attention is restricted only to the case where $d = 2, 3$. For $\rho > 0$ and $x \in \mathbb{R}^d$, denote $B_\rho(x)$ the d-dimensional open ball of radius ρ and center x, i.e.,

$$B_\rho(x) = \left\{y \in \mathbb{R}^d\colon\ |x - y| < \rho\right\}.$$

In particular, denote $B_\rho(0) \equiv B_\rho$. Furthermore, let $\omega_d = \frac{2\pi^{\frac{d}{2}}}{\gamma(\frac{d}{2})}$ denote the surface of the unit sphere in $\mathbb{R}^d$, where γ is the Gamma function.

Let Ω be a smooth domain in $\mathbb{R}^d$ with the boundary Γ. The classical space $L^p(\Omega)$ contains all of the p-integrable complex-valued functions over Ω equipped with the norm:

$$\|f\|_{L^p} = \left(\int_\Omega \left|f(x)\right|^p dx\right)^{\frac{1}{p}}.$$

We use the notation $H^s(\Omega)$ for the Sobolev space of order $s > 0$ in the usual sense. For $s \in \mathbb{N}$, the norm for the Sobolev space $H^s(\Omega)$ takes the following form:

$$\|f\|_s = \sum_{|\alpha| \leqslant s} \left\|\partial_x^\alpha f\right\|_{L^2}.$$

Let $t_0 \in \mathbb{R}^d$, $p \in \mathbb{N}$ and $\rho > 0$. The Sobolev space $L^q(B_\rho(t_0), H^s(\Omega))$ is defined by

$$L^q\big(B_\rho(t_0), H^s(\Omega)\big) = \left\{\varphi(t,x)\colon\ t \to \left\|\varphi(t,.)\right\|_s \in L^q\big(B_\rho(t_0)\big)\right\},$$

along with the norm

$$\|f\|_{q,s} = \left(\int_{B_\rho(t_0)} \left\|\varphi(t,.)\right\|_s^q dt\right)^{\frac{1}{q}}.$$

Moreover, the notation $\|f\|_{L^p(\Gamma)}$ is used when the norm $\|f\|_{L^p}$ is taken on Γ.

The Hölder space $C^0(B_\rho, L^p(\Gamma))$ is defined by

$$C^0\big(B_\rho(t_0), L^p(\Gamma)\big) = \big\{\varphi(t,x) : t \to \big\|\varphi(t,.)\big\|_{L^p(\Gamma)} \in C^0\big(B_\rho(t_0)\big)\big\},$$

whose norm is given by

$$\|f\|_{C^0(B_\rho(t_0), L^p(\Gamma))} = \sup_{t\in B_\rho(t_0)} \big\|\varphi(t,.)\big\|_{L^p(\Gamma)}.$$

Finally, define the Fourier transform of $g(x) \in L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(g)(\xi) := \frac{1}{(2\pi)^d} \int\limits_{B_{R_0}} g(x) e^{-i\xi.x}\, dx.$$

2. The multi-frequency inverse source problem

2.1. Formulation of the multi-frequency inverse source problem

Consider in $\mathbb{R}^d$ the inverse source problem (ISP) of determining an unknown scalar source S to the homogeneous Helmholtz equation

$$\Delta\psi + k^2\psi = S(x), \tag{2.1}$$

where k is the wavenumber of the radiated scalar field ψ. Assume that the source function $S(x) \in L^2(\mathbb{R}^d)$ has a compact support V_0. It is further required that the radiated field ψ satisfies the Sommerfeld radiation condition:

$$\partial_r\psi - ik\psi = o\big(r^{\frac{1-d}{2}}\big), \tag{2.2}$$

as r goes to infinity. Also, since ψ depends on the frequency k, we sometimes employ $\psi(k,x)$ in place of $\psi(x)$ to emphasize the dependence.

Define the Green function in the whole space

$$\mathcal{G}(k,r) = \begin{cases} -\frac{i}{4} H_0^{(1)}(kr) & \text{if } d = 2, \\ -\frac{e^{ikr}}{4\pi r} & \text{if } d = 3, \end{cases}$$

where $H_0^{(1)}(kr)$ is the Hankel function of first kind with order 0. Then, there is a unique solution $\psi(x)$ satisfying the Helmholtz equation (2.1) and the radiation condition (2.2). In addition,

$$\psi(x) = \int\limits_{\mathbb{R}^d} \mathcal{G}\big(k, |x-y|\big) S(y)\, dy.$$

Having defined the forward problem, we can now introduce the inverse source problem (ISP). For simplicity, we assume that there exists a positive real R_0 such that $\overline{V}_0 \subset B(R_0) \subset \Omega$. The multi-frequency ISP consists of recovering the source function from the measurement of the radiated wave $\psi(k,x)$ on the boundary Γ for all $k \in [0,k_0]$. It may be stated as: *to reconstruct the source function S from the measured field $\psi(k,x)$ for all $k \in [0,k_0]$, $x \in \Gamma$.*

2.2. Ill-posedness of the ISP at a fixed frequency

For a function S in $L^2(B_{R_0})$, we introduce the scattering operator

$$L(k)S(x) = \int_{B_{R_0}} \mathcal{G}\big(k, |x-y|\big) S(y)\, dy, \quad x \in \Gamma. \tag{2.3}$$

The ISP at a fixed frequency may also be formulated as follows: to find the solution S to the linear equation $L(k)S(x) = \psi(k,x)$, $x \in \Gamma$.

We further study the ill-posedness of the ISP at fixed frequency based on the analysis of the spectrum of the scattering operator $L(k)$. For the sake of simplicity, our study is restricted to the two-dimensional case, i.e., $d = 2$. It is also assumed that the measurements are taken on a circle of the disk $\Omega = B_R$. Here R is large enough so that $R_0 < R$ or the support of the source function is contained in Ω. Note that although the mathematical formulation depends on the surface where the measurements are taken, the nature of the ISP and particularly the ill-posedness remains the same.

We begin with some useful properties of the operator $L(k)$.

Proposition 2.1. *The following results hold*:

(i) *The scattering operator $L(k)$ is compact from $L^2(B_{R_0})$ to $L^2(\partial B_R)$ with the following singular value decomposition*

$$L(k) = \sum_{l \geqslant 0} \sigma_l P_l,$$

where $P_l = \langle ., \psi_l \rangle \phi_l + \langle ., \psi_{-l} \rangle \phi_{-l}$ and

$$\begin{cases} \phi_l(\theta) = \dfrac{1}{\sqrt{2\pi}} e^{il\theta}, \\ \sigma_l(k) = \dfrac{\pi}{2} \big| H_{|l|}^{(1)}(kR) \big| \left(\displaystyle\int_0^{R_0} J_l^2(kr) r\, dr \right)^{\frac{1}{2}}, \\ \psi_l(r,\theta) = \dfrac{1}{\sqrt{2\pi} \big(\int_0^{R_0} J_l^2(kr) r\, dr \big)^{\frac{1}{2}}} J_l(kr) e^{il\theta}. \end{cases}$$

(ii) *The null space of L is orthogonal to $\{\psi_l\colon\ l \geqslant 0\}$ in $L^2(B_{R_0})$.*

Proof. By elliptic regularity it can be shown that $\psi(x)$ lies in $H^2(B_R)$. Far away from the compact support V_0 of the source function, ψ is C^∞ (the kernel $\mathcal{G}$ is C^∞). Hence the trace ψ on Γ belongs to $H^s(\partial B_R)$ for all $s \geqslant 0$. By compact embedding of Sobolev spaces on bounded domains, we deduce the compactness of the operator $L(k)$.

By the Graf addition formula, we obtain a series representation of the Green function

$$\mathcal{G}\big(k, |x-y|\big) = -\frac{i}{4} \sum_{l \in \mathbb{Z}} J_l(kr_x) H_l^{(1)}(kr_y) e^{il(\theta_y - \theta_x)},$$

where (r_x, θ_x) and (r_y, θ_y) are the polar coordinates of x and y satisfying $r_x < r_y$. By substituting the kernel with its series representation in (2.3), we obtain

$$L(k)S(\theta) = -\frac{i}{4}\gamma_R \int_0^{2\pi}\int_0^{R_0} \sum_{l\in\mathbb{Z}} J_l(k\rho)H_l^{(1)}(kr)e^{il(\theta-\beta)}S(\rho,\beta)\rho\,d\rho\,d\beta.$$

Due to the smoothness of the kernel $\mathcal{G}$ on the circle ∂B_R, an application of the convergence dominated theorem yields

$$L(k)S(\theta) = -\frac{i}{4}\sum_{l\in\mathbb{Z}} H_l^{(1)}(kR)e^{il\theta}\int_0^{2\pi}\int_0^{R_0} J_l(k\rho)e^{-il\beta}S(\rho,\beta)\rho\,d\rho\,d\beta.$$

For a fixed g in $L^2(\partial B_R)$, the adjoint operator of the operator L-adjoint takes the following form

$$L(k)^*g(\theta) = \frac{i}{4}\sum_{l\in\mathbb{Z}} \overline{H_l^{(1)}(kR)}\,J_l(k\rho)e^{-il\theta}\int_0^{2\pi} g(\beta)e^{il\beta}\,d\beta.$$

It follows for g in $L^2(\partial B_R)$ that

$$L(k)L(k)^*g(\theta) = \frac{\pi}{8}\sum_{l\in\mathbb{Z}} \left|H_l^{(1)}(kR)\right|^2 \int_0^{R_0} J_l^2(k\rho)\rho d\rho \int_0^{2\pi} g(\beta)e^{-il\beta}\,d\beta\,e^{il\theta}.$$

The proof of (ii) is now completed by using that the kernel of the operator L is the orthogonal complement of the range of L^*. $\square$

Remark 2.1. In fact, the space $\text{span}\{\psi_l\colon\ l \geqslant 0\}$ is identical to the spaces $\text{span}\{e^{ixk.d}\colon\ d \in \partial B_1\}$ and $\{u(x) \in H^2(B(R_0))\colon\ \Delta u + k^2u = 0\}$ in $L^2(B(R_0))$.

Based on this remark, it is obvious to construct a function in the kernel of L. Actually, it suffices to choose a function $h(r)$ defined on $(0, R_0)$ that is orthogonal to $J_l(kr)$ in $L^2(B_{R_0})$ for a fixed $l \geqslant 0$. For example, the function $h(r) = 1 - \langle 1, J_l(kr)\rangle_0$. Then the function $h(r)e^{il\theta}$ belongs to the kernel of $L(k)$, which explains the non-uniqueness of the ISP. The sources in the kernel of L, known as *non-radiating sources*, produce fields that are identically zero outside their source volume B_{R_0}.

The fact that there are non-radiating source functions makes it impossible to determine the exact details of a source from measurements taken outside B_{R_0} unless additional constraints are imposed on the source. The constraints must be sufficient to specify which one of the infinite number of solutions to the ISP is of particular interest. An often used constraint in practice is to minimize the source energy [18]:

$$\int_{B_{R_0}} \left|S(x)\right|^2 dx.$$

Next, we present an explicit minimum energy solution to the ISP. There exists a unique function $\tilde{S}$ in $L^2(B_{R_0})$ [15] satisfying

$$\int_\Gamma \left(L(k)\tilde{S} - \psi\right)^2 d\sigma_x = \min_{P\in L^2(B_{R_0})} \int_\Gamma \left(L(k)P - \psi\right)^2 d\sigma_x,$$

where $\tilde{S}$ is the minimum energy solution to the ISP at fixed frequency, which is given by

$$\tilde{S} = \sum_{l \geqslant 0} \frac{P_l^* P}{\sigma_l^2(k)} L^*(k)\psi. \tag{2.4}$$

Here $P_l^* = \langle ., \phi_l \rangle \psi_l + \langle ., \phi_{-l} \rangle \psi_{-l}$, the adjoint operator of P_l.

We remark from (2.4) that $\tilde{S}$ is orthogonal to the nullspace of $L(k)$. Thus the information on the real source S encoded in the nullspace of $L(k)$ is lost, which leads to the lack of uniqueness. Using the asymptotics of Hankel and Bessel functions for large argument [1], it is evident to see that the singular eigenvalue σ_l decreases exponentially as the index l increases, which illustrates the ill-posedness of the ISP at fixed frequency.

3. Uniqueness and stability estimates

In order to overcome the ill-posedness and non-uniqueness of the ISP, we propose to take measurements at multiple frequencies. This section is devoted to the mathematical study of fundamental issues on uniqueness and stability for the ISP. We first prove the uniqueness of the multi-frequency inverse source problem when the data is known for $k \in [0, k_0]$. We also investigate the stability of the ISP. The first stability result deals with the case where the frequency k_0 is fixed but the source radius R_0 is variable. The second one fixes R_0 and leaves the frequency k_0 free. The main results are Theorems 3.2 and 3.3, and Corollaries 3.1 and 3.2.

3.1. Uniqueness for the ISP

Let $\xi \in \mathbb{R}^d$ such that $|\xi| = k$ and $\nu(x)$ be the outward normal derivative on Γ. Multiplying Eq. (2.1) by $e^{-i\xi.x}$ and integrating over Ω, we obtain

$$\mathcal{F}(S)(\xi) = \int_{\Gamma} e^{-i\xi.x}\big(\partial_\nu \psi(k,x) + i\xi.\nu\psi(k,x)\big)\, d\sigma_x, \quad |\xi| = k \in [0, k_0]. \tag{3.1}$$

It is evident from the above formula that by collecting the measurements $\psi(k,x)$ on a band of frequency $[0, k_0]$, the Fourier transform of S on B_{k_0} can be reconstructed directly. In addition, our next result indicates that this amount of information is sufficient to determine the source function S uniquely.

Theorem 3.1. *Let $(k_j)_j$ be a set of real numbers with an accumulation point. Then the measurements $(\psi(k_j, .))_j$ on Γ determine uniquely the source function S.*

Proof. Assume that two source functions S_1 and S_2 produce the same data $(\psi(k_j, .))_j$ on Γ, and set $S = S_1 - S_2$. Then Eq. (3.1) implies that

$$\int_{B_{R_0}} S(x) e^{-i\xi.x}\, dx = 0, \quad \forall \xi \in \partial B_{k_j}.$$

Since B_{R_0} is a bounded domain, the integral above is an entire function of $\xi \in \mathbb{C}^d$. In particular, the integral vanishes on a set of complex numbers $\{\xi_j\} \in \mathbb{C}^d$, such that $\xi_j^{(1)} = k_j$ and $\xi_j^{(l)} = 0$ if $l \neq 1$ with an accumulation point. Then from unique continuation [24], $\mathcal{F}(S)(\xi)$ is zero on the whole complex domain $\mathbb{C}^d$. Since the Fourier transform is invertible, $S(x)$ also vanishes identically on $\mathbb{R}^d$. □

The following remarks about the usefulness of the uniqueness result are in order.

Remark 3.1. In fact $\mathcal{F}(S)(\xi)$, $|\xi| < k_0$ represents the low-frequency modes of the source function S. Many questions arise from Eq. (3.1): Do the boundary measurements $\psi(k,.)$ contain further details about the source? More precisely, could one extract the high frequency contents of S? If the answer is yes, then the source function can be approximated with arbitrary accuracy. In fact, in the absence of noise, theoretically the construction resolution may be increased without a limit as suggested by the above uniqueness theorem. This conclusion obviously contradicts to the uncertainty principle according to which features smaller than one half of the wavelength might not be resolvable. In reality, since some noise is always present in any data set, there is a limit to the possible resolution. Therefore, it is critically important to address the stability for the multi-frequency ISP.

Remark 3.2. From Eq. (3.1), we may outline the general steps for reconstructing the function S. First, it should be remarked that $\partial_\nu \psi(k,.)$ can be directly computed from $\psi(k,.)$ by using the Dirichlet-to-Neumann operator $\Lambda(k)$. The well-known properties of the operator $\Lambda(k)$ assure a stable reconstruction of $\mathcal{F}(S)$ [11]. Hence the ill-posedness of the inverse problem is primarily attributed to the reconstruction of S from the knowledge of its low-frequency modes:

$$f(\xi) := \frac{1}{(2\pi)^d} \int\limits_{B_{R_0}} S(x) e^{-i\xi.x}\, dx, \quad \forall \xi \in B_{k_0}. \tag{3.2}$$

The ISP can be reformulated as follows: Given R_0, k_0 and $f(\xi)$, $\xi \in B_{k_0}$, find S.

Remark 3.3. A similar uniqueness result is derived in [21] but for a chosen unbounded set of frequencies (the Dirichlet eigenvalues of the Laplacian). Our uniqueness result corresponds to the applications mentioned in the introduction, and the frequency continuation method developed in [5–8,10,11] where the measurements are taken on a bounded band of frequency.

In the rest of this section, we establish stability estimates for the multi-frequency ISP with respect to the dimensionless number $c = k_0 R_0$. We distinguish two different cases: (1) The highest wave number k_0 is fixed but the support of $S(x)$ is variable (R_0). (2) The radius of the support R_0 is fixed but k_0 is a variable.

3.2. Stability for the ISP. Part I: The stability estimate for a variable compact support

The behavior of the singular eigenvalues σ_j of the linear operator $S \to f$, as a function of c is derived in [29,28]. Basically, σ_j tend to one when c approaches zero and decrease exponentially with respect to j when c is large. This shows that the ISP is well-posed when c is small and becomes extremely ill-posed when c is large. Without loss of generality, assume that $k_0 = 1$. Next, we reconstruct $S(x)$, a solution to (3.2), from a given function $f(\xi)$, $\xi \in B_1$ and constant c. Our derivation of an approximation of the source function follows the general idea of Ramm [27] who derived a general analytic inversion formula for the Fourier transform of a compactly supported function from a compact set. Using the analytic formula and under a smoothness assumption on S we will first derive an analytic approximation to the source function.

Let $h(x)$ be a normalized cut-off function: $h \in C_0^\infty(B_1)$ and $\frac{1}{(2\pi)^d}\int_{B_1} h(x)\, dx = 1$. In addition, we assume that h is radially symmetric. A classical example is the following:

$$h(x) = \begin{cases} \lambda_0 e^{\frac{1}{|x|^2-1}}, & |x| \leqslant 1, \\ 0, & |x| \geqslant 1, \end{cases} \tag{3.3}$$

where $\lambda_0^{-1} := \frac{\omega_d}{(2\pi)^d}\int_{-1}^{1} e^{\frac{1}{r^2-1}} r^{d-1}\, dr$. An analytic approximation of the source function S is given by

$$S_N(x) := \int_{B_c} \delta_N(x-y) S(y)\,dy = \frac{1}{(2\pi)^d} \int_{B_1} h_N(\xi) f(\xi) e^{i\xi.x}\,d\xi, \tag{3.4}$$

where

$$h_N(x) = \int_{\mathbb{R}^d} \delta_N(\xi) e^{-i\xi.x}\,d\xi \tag{3.5}$$

and

$$\delta_N(x) := \left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}} \left(1 - \frac{\delta^2 |x|^2}{c^2}\right)^{N-1} \mathcal{F}^{-1}(h)(x), \quad N \geqslant 1, \tag{3.6}$$

with δ a fixed constant satisfying $0 < \delta < \frac{1}{2}$.

Note that since $\mathcal{F}^{-1}(h)(x)$ is an entire function and the factor $(1 - \frac{\delta^2|x|^2}{c^2})^{N-1}$ is a polynomial, $\delta_N(x)$ is also an entire function on $\mathbb{C}^d$. In addition, from Appendix A, $h_N(x)$, the Fourier transform of δ_N, is radially symmetric and compactly supported in B_1.

The following lemma was proved in [27].

Lemma 3.1. *The function $\mathcal{F}^{-1}(h)(x)$ is an entire function of 1-exponential type. Then, there exists a strictly positive constant $C > 0$ such that*

$$\left|\mathcal{F}^{-1}(h)(z)\right| \leqslant C e^{-|z|}, \quad \forall z \in \mathbb{C}^d.$$

We are now ready to present the first stability estimate.

Lemma 3.2. *Let S be a function in $C^1(\overline{B}_c)$ and δ be fixed in $(0, \frac{1}{2})$. If*

$$\|S\|_{C^1(\overline{B}_c)} \leqslant M,$$

then

$$\|S - S_N\|_{C^0(\overline{B}_c)} \leqslant C_d \frac{c}{\delta} \frac{M}{N^{\frac{1}{2}}}, \quad \textit{as } N \to \infty,$$

where M is a positive constant and C_d is a constant depending only on the dimension d.

Proof. Observe that $\mathcal{F}^{-1}(h)(0) = \int_{B_1} h(\xi)\,d\xi = 1$. Since $h \in C_0^\infty(B_1)$, the function $\mathcal{F}^{-1}(h)(\xi)$ is an entire function on $\mathbb{C}^d$. Hence $\delta_N(\xi)$, the product of a polynomial function $(1 - \frac{\delta^2|x|^2}{c^2})^{N-1}$ and $\mathcal{F}^{-1}(h)(\xi)$, is also an entire function. Thus, the convolution of $\delta_N(\xi)$ with S is well defined and belongs to $C^0(\overline{B}_c)$. Let x be fixed in B_c. A simple calculation yields

$$S_N(x) = \left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}} \int_0^{2c} \left(1 - \frac{\delta^2 r^2}{c^2}\right)^{N-1} \psi_x(r) r^{d-1}\,dr, \tag{3.7}$$

where

$$\psi_x(r) := \int_{\partial B_1} \mathcal{F}^{-1}(h)(r\omega) S(x - r\omega)\, d\omega.$$

The C^1 function $\psi_x(r)$ is compactly supported in $[0, 2c]$. For simplicity, let us treat only the case $d = 2$.

Integration by parts in the expression (3.7) yields

$$S_N(x) = S(x) + \frac{1}{2\pi} \int_0^{2c} \left(1 - \frac{\delta^2 r^2}{c^2}\right)^N \psi_x'(r)\, dr,$$

where

$$\psi_x'(r) = \int_{\partial B_1} \left(\nabla \mathcal{F}^{-1}(h)(r\omega).\omega S(x - r\omega) - \mathcal{F}^{-1}(h)(r\omega) \nabla S(x - r\omega).\omega\right) d\omega.$$

Therefore

$$\left|S_N(x) - S(x)\right| \leqslant 2Mc\left(\left\|\nabla \mathcal{F}^{-1}(h)\right\|_{C^0(\overline{B_{2c}})} + 1\right) \int_0^1 \left(1 - 4\delta^2 r^2\right)^N dr.$$

It is easily seen that $\|\nabla \mathcal{F}^{-1}(h)\|_{C^0(\overline{B_{2c}})} \leqslant 1$ for all $\xi \in \mathbb{R}^2$.

Consequently,

$$\left|S_N(x) - S(x)\right| \leqslant 4cM \int_0^1 \left(1 - 4\delta^2 r^2\right)^N dr.$$

On the other hand,

$$\left(1 - 4\delta^2 r^2\right)^N \leqslant e^{-4\delta^2 N r^2}$$

for all $r \in [0, 1]$.

Hence

$$\int_0^1 \left(1 - 4\delta^2 r^2\right)^N dr \leqslant \int_0^1 e^{-4\delta^2 N r^2}\, dr \leqslant \left(2\pi \int_0^1 e^{-4\delta^2 N r^2} r\, dr\right)^{\frac{1}{2}} = \left(\frac{\pi}{2\delta^2 N}\left(1 - e^{-4\delta^2 N}\right)\right)^{\frac{1}{2}}.$$

By combining the above estimates, we obtain

$$\left|S_N(x) - S(x)\right| \leqslant \frac{(8\pi)^{\frac{1}{2}}}{\delta} Mc \frac{1}{N^{\frac{1}{2}}}, \quad \forall x \in B_c.$$

The proof for the three-dimensional case ($d = 3$) may be given similarly by using the Watson Lemma below. $\square$

Lemma 3.3 *(Watson's Lemma). (See [12].) If $g(t)$ is continuous on $[0,1]$ and has an asymptotic expansion*:

$$g(t)=t^{\alpha}\sum_{n=0}^{+\infty}g_n t^{\beta_n},\quad \text{as } t\to 0,$$

with $(\beta_n)>0$ an increasing sequence and $\alpha>0$, then

$$\int_0^1 e^{-xt}g(t)\,dt\sim\sum_{n=0}^{+\infty}g_n\frac{\Gamma(\alpha+\beta_n+1)}{x^{\alpha+\beta_n+1}},\quad \text{as } x\to+\infty,$$

where $\Gamma(z)$ is the Gamma function.

We next present an intermediate stability result for the multi-frequency ISP.

Lemma 3.4. *Let S be a function in $C^1(\overline{B}_c)$, δ be a constant satisfying $0<\delta<\frac{1}{2}$ and f is the Fourier transform defined by* (3.2)*. If*

$$\|S\|_{C^1(\overline{B}_c)}\leqslant M,$$

then

$$\|S\|_{C^0(\overline{B}_c)}\leqslant C_1\frac{c}{\delta}\frac{M}{N^{\frac{1}{2}}}+C_2\left(1+\left(\frac{\delta^2}{c^2}N\right)^{\frac{d+1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}\right)\|f\|_{C^0(\overline{B}_1)},$$

for all $N\geqslant 1$, where M is a positive constant, C_1 and C_2 are constants depending only on the dimension d.

Proof. Clearly

$$\|S\|_{C^0(\overline{B}_c)}\leqslant\|S_N\|_{C^0(\overline{B}_c)}+\|S-S_N\|_{C^0(\overline{B}_c)}.$$

Recall that

$$S_N(x):=\frac{1}{(2\pi)^d}\int_{B_1}h_N(\xi)f(\xi)e^{-i\xi.x}\,d\xi.$$

Thus

$$\|S_N\|_{C^0(\overline{B}_c)}\leqslant\frac{1}{(2\pi)^d}\big\|h_N(\xi)\big\|_{L^1(\mathbb{R}^d)}\|f\|_{C^0(\overline{B}_1)}. \tag{3.8}$$

From (3.5), it follows that $\|h_N\|_{L^1(\mathbb{R}^d)}\leqslant\omega_d\|\delta_N\|_{L^1(\mathbb{R}^d)}$, i.e.,

$$\|h_N\|_{L^1(\mathbb{R}^d)}=\omega_d\left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d}\left|1-\frac{\delta^2|x|^2}{c^2}\right|^{N-1}\big|\mathcal{F}^{-1}(h)(x)\big|\,dx.$$

On the other hand, Lemma 3.1 implies

$$\left|\mathcal{F}^{-1}(h)(x)\right| \leqslant Ce^{-|x|}, \quad \forall x \in \mathbb{R}^d.$$

Hence

$$\begin{aligned}\|h_N\|_{L^1(\mathbb{R}^d)} &\leqslant C\omega_d\left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}} \int\limits_{\mathbb{R}^d} \left|1 - \frac{\delta^2|x|^2}{c^2}\right|^{N-1} e^{-|x|}\,dx \\ &\leqslant C\omega_d^2\left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}} \int\limits_0^{+\infty} \left|1 - \frac{\delta^2 r^2}{c^2}\right|^{N-1} e^{-r} r^{d-1}\,dr.\end{aligned}$$

The last term may be decomposed into two terms:

$$\int\limits_0^{+\infty} \left|1 - \frac{\delta^2 r^2}{c^2}\right|^{N-1} e^{-r} r^{d-1}\,dr = I_1 + I_2,$$

where

$$\begin{aligned}I_1 &:= \int\limits_0^{\sqrt{2}\frac{c}{\delta}} \left|1 - \frac{\delta^2 r^2}{c^2}\right|^{N-1} e^{-r} r^{d-1}\,dr, \\ I_2 &:= \int\limits_{\sqrt{2}\frac{c}{\delta}}^{+\infty} \left(\frac{\delta^2 r^2}{c^2} - 1\right)^{N-1} e^{-r} r^{d-1}\,dr.\end{aligned}$$

By a change of variables, we get

$$I_1 = \frac{c^d}{\delta^d} \int\limits_0^{\sqrt{2}} \left|1 - r^2\right|^{N-1} e^{-\frac{c}{\delta}r} r^{d-1}\,dr$$

or

$$I_1 = \frac{c^d}{\delta^d} \int\limits_0^1 \left(1 - r^2\right)^{N-1} e^{-\frac{c}{\delta}r} r^{d-1}\,dr + \frac{c^d}{\delta^d} \int\limits_1^{\sqrt{2}} \left(r^2 - 1\right)^{N-1} e^{-\frac{c}{\delta}r} r^{d-1}\,dr.$$

By using the following simple inequalities:

$$\begin{aligned}&\left(1 - r^2\right)^{N-1} \leqslant e^{-(N-1)r^2}, \quad \forall r \in (0, 1), \\ &\left(r^2 - 1\right)^{N-1} \leqslant e^{-(N-1)(2-r^2)}, \quad \forall r \in (0, \sqrt{2}),\end{aligned}$$

we arrive at

$$I_1 \leqslant \frac{c^d}{\delta^d}\left(\int_0^1 e^{-(N-1)r^2} r^{d-1}\,dr + \int_1^{\sqrt{2}} e^{-(N-1)(2-r^2)} r^{d-1}\,dr\right).$$

Thus

$$I_1 \leqslant \frac{c^d}{\delta^d}\frac{C}{N^{\frac{d}{2}}},$$

for some constant C that depends only on the dimension.

A simple calculation yields further that

$$\begin{aligned} I_2 &\leqslant \left(\frac{\delta}{c}\right)^{2(N-1)} \int_{\sqrt{2}\frac{c}{\delta}}^{+\infty} r^{2(N-1)+d-1} e^{-r}\,dr \\ &\leqslant \left(\frac{\delta}{c}\right)^{2(N-1)} \int_0^{+\infty} r^{2(N-1)+d-1} e^{-r}\,dr \\ &\leqslant \left(\frac{\delta}{c}\right)^{2N-2} (2N+d-3)!. \end{aligned}$$

Hence

$$\int_0^{+\infty} \left|1 - \frac{\delta^2 r^2}{c^2}\right|^{N-1} e^{-r} r^{d-1}\,dr \leqslant \frac{c^2}{\delta^2}\frac{C}{N^{\frac{d}{2}}} + \left(\frac{\delta}{c}\right)^{2N-2}(2N+d-3)!.$$

By the Stirling approximation [1]

$$(2N+d-3)! \leqslant e(2\pi)^{\frac{1}{2}}(2N+d-3)^{\frac{1}{2}}\left(\frac{2N+d-3}{e}\right)^{2N+d-3},$$

we obtain

$$\int_0^{+\infty} \left|1 - \frac{\delta^2 r^2}{c^2}\right|^{N-1} e^{-r} r^{d-1}\,dr \leqslant \frac{c^2}{\delta^2}\frac{C}{N^{\frac{d}{2}}} + e(2\pi)^{\frac{1}{2}}\frac{\delta}{c}N^{\frac{1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}.$$

Therefore

$$\|h_N\|_{L^1(B_1)} \leqslant (2\pi)^d C_2\left(\frac{\delta^2}{c^2}N\right)^{\frac{d}{2}}\left(\frac{c^2}{\delta^2}\frac{1}{N^{\frac{d}{2}}} + \frac{\delta}{c}N^{\frac{1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}\right),$$

which leads to the final estimate for h_N

$$\|h_N\|_{L^1(B_1)} \leqslant (2\pi)^d C_2\left(1 + \left(\frac{\delta^2}{c^2}N\right)^{\frac{d+1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}\right).$$

By substituting the above estimate into (3.8), we deduce that

$$\|S_N\|_{C^0(\overline{B}_c)} \leqslant C_2\left(1+\left(\frac{\delta^2}{c^2}N\right)^{\frac{d+1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}\right)\|f\|_{C^0(\overline{B}_1)}, \tag{3.9}$$

for all $N \geqslant 1$.

The proof is completed by combining Lemma 3.2 and the estimate (3.9). $\square$

The following is the first of our two main results on the stability of the multi-frequency ISP.

Theorem 3.2. *Let S be a function in $C^1(\overline{B}_c)$ and δ be a constant satisfying $0 < \delta < \frac{1}{2}$. Assume that*

$$\epsilon = \|\partial_\nu \psi\|_{C^0(B_1, L^1(\Gamma))} + \|k\psi\|_{C^0(B_1, L^1(\Gamma))} < 1$$

and

$$\|S\|_{C^1(\overline{B}_c)} \leqslant M,$$

where M is a positive constant. The following statements hold:

A) *If $c \leqslant \epsilon$, then*

$$\|S\|_{C^0(\overline{B}_c)} \leqslant M\epsilon.$$

B) *If $c \geqslant \frac{1}{2}\epsilon^{\frac{1}{1+2d}}$ and $\beta \in (0,1)$, then*

$$\|S\|_{C^0(\overline{B}_c)} \leqslant C\left(M\left(1+\frac{c^\beta}{\delta^\beta}\right)^{\frac{1}{2(1+\beta)}}\left(\frac{c}{\delta}\right)^{\frac{1}{(1+\beta)}}+1\right)\frac{1}{(\ln(\epsilon^{-1}))^{\frac{1}{2(1+\beta)}}},$$

where C is a constant that only depends on the dimension d and β.

Proof. From the assumption, it is clear that $\|f\|_{C^0(\overline{B}_1)} \leqslant \epsilon$.

A) Since $S(x)$ belongs to $C^1(\overline{B}_c)$, we have

$$\|S\|_{C^0(\overline{B}_c)} \leqslant Mc.$$

By noting the assumption $c \leqslant \epsilon$, Part A) is proved.

B) For N large, the stability estimate of Lemma 3.4 can be rewritten as

$$\|S\|_{C^0(\overline{B}_c)} \leqslant C_1\frac{c}{\delta}\frac{M}{N^{\frac{1}{2}}} + C_2\|f\|_{C^0(\overline{B}_1)} + C_2 e^{A_\gamma}\|f\|^{1-\gamma}_{C^0(\overline{B}_1)},$$

where

$$A_\gamma := \frac{d+1}{2}\ln\left(\frac{\delta^2}{c^2}N\right) + 2N\ln\left(\frac{2\delta N}{ec}\right) - \gamma\ln\left(\|f\|^{-1}_{C^0(\overline{B}_1)}\right).$$

Let β be a fixed real in $(0,1)$ and set $N=(\lambda^{-1}\ln(\|f\|^{-\gamma}_{C^0(\overline{B}_1)}))^{\frac{1}{1+\beta}}$, where λ is a positive constant. In fact, λ may be chosen so that $A_\gamma \leqslant 0$, which implies that

$$\|S\|_{C^0(\overline{B}_c)} \leqslant C_1\frac{c}{\delta}\frac{M}{N^{\frac{1}{2}}}+2C_2\|f\|^{1-\gamma}_{C^0(\overline{B}_1)}.$$

It follows from

$$\ln(t)\leqslant\frac{t^\beta}{e\beta},\quad \forall t\geqslant 1,$$

that

$$A_\gamma\leqslant\left(\frac{\lambda^{-1}}{e\beta}\left(\frac{d+1}{2}\left(\frac{\delta}{c}\right)^{2\beta}+2\left(\frac{2\delta}{ec}\right)^{\beta}\right)-1\right)\ln\bigl(\|f\|^{-\gamma}_{C^0(\overline{B}_1)}\bigr).$$

By taking $\lambda=\lambda_c:=\frac{1}{e\beta}(\frac{d+1}{2}(\frac{\delta}{c})^{2\beta}+2(\frac{2\delta}{ec})^{\beta})$, $\|f\|_{C^0(\overline{B}_1)}<1$, and $\gamma=\frac{1}{2}$, we finally obtain

$$\|S\|_{C^0(\overline{B}_c)}\leqslant\bigl(2^{\frac{1}{2(1+\beta)}}C_1Mc\lambda_c^{\frac{1}{2(1+\beta)}}+C_2\bigr)\frac{1}{(\ln(\|f\|^{-1}_{C^0(\overline{B}_1)}))^{\frac{1}{2(1+\beta)}}},$$

which completes the proof of Part B). $\square$

For the sake of clarity we give the following corollary (corresponds to the results of Theorem 3.2 with $\delta=\frac{1}{4}$ and $\beta=\frac{1}{2}$).

Corollary 3.1. *Let S be a function in $C^1(\overline{B}_c)$. Assume that*

$$\epsilon=\|\partial_\nu\psi\|_{C^0(B_1,L^1(\Gamma))}+\|k\psi\|_{C^0(B_1,L^1(\Gamma))}<1$$

and

$$\|S\|_{C^1(\overline{B}_c)}\leqslant M,$$

where M is a positive constant. The following statements hold:

A) *If $c\leqslant\epsilon$, then*

$$\|S\|_{C^0(\overline{B}_c)}\leqslant M\epsilon.$$

B) *If $c\geqslant\frac{1}{2}\epsilon^{\frac{1}{1+2d}}$, then*

$$\|S\|_{C^0(\overline{B}_c)}\leqslant C\bigl(M(1+2\sqrt{c})^{\frac{1}{3}}(4c)^{\frac{2}{3}}+1\bigr)\frac{1}{(\ln(\epsilon^{-1}))^{\frac{1}{3}}},$$

where C is a constant that only depends on the dimension d.

The results obtained in Theorem 3.2 and Corollary 3.1 show clearly that for the measurement $\psi(k,x)|_\Gamma$ for $k\in[0,1]$ with a noise ϵ, the recovery of the source function is linearly stable if its compact support radius c is smaller than ϵ but is logarithmic stable if $c\geqslant\epsilon$.

3.3. Stability for the ISP. Part II: The stability estimate on a variable frequency band

We next study the stability of the ISP with respect to c, when R_0 is fixed. For simplicity, we assume that $R_0 = 1$. Recall that the function

$$f(\xi) = \int_\Gamma e^{-i\xi.x}\big(\partial_\nu \psi(k,x) + i\xi.\nu\psi(k,x)\big)\,d\sigma_x$$

is known on B_c.

Let $h(x)$ be the normalized cut-off function introduced in the previous subsection. We denote $h_c(x) := \frac{1}{c^d}h(\frac{x}{c})$ and define

$$S_N(x) := \int_{B_1} \delta_N(x-y)S(y)\,dy = \frac{1}{(2\pi)^d}\int_{B_1} h_N(\xi)f(\xi)e^{i\xi.x}\,d\xi,$$

where

$$h_N(x) = \int_{\mathbb{R}^d} \delta_N(\xi)e^{-i\xi.x}\,d\xi$$

and

$$\delta_N(\xi) := \left(\frac{\delta^2 N}{\pi}\right)^{\frac{d}{2}}\big(1-\delta^2|\xi|^2\big)^{N-1}\mathcal{F}^{-1}(h_c)(\xi), \quad N \geqslant 1, \tag{3.10}$$

with $\delta \in (0, \frac{1}{2})$ a fixed constant.

Lemma 3.5. *Let S be a function in $C^1(\overline{B}_1)$ and δ be fixed in $(0, \frac{1}{2})$. If*

$$\|S\|_{C^1(\overline{B}_1)} \leqslant M,$$

then

$$\|S - S_N\|_{C^0(\overline{B}_1)} \leqslant \frac{C_d}{\delta^{\frac{1}{2}}}M\left(1+\frac{1}{c}\right)\frac{1}{N^{\frac{1}{4}}}, \quad \text{as } N \to \infty,$$

where M is a positive constant and C_d is a constant depending only on the dimension d.

Proof. For a point x in B_1, a simple calculation yields

$$S_N(x) = \left(\frac{\delta^2 N}{\pi}\right)^{\frac{d}{2}}\int_0^2 \big(1-\delta^2 r^2\big)^{N-1}\psi_x(r)r^{d-1}\,dr, \tag{3.11}$$

where

$$\psi_x(r) := \int_{\partial B_1} \mathcal{F}^{-1}(h_c)(r\omega)S(x-r\omega)\,d\omega.$$

The function $\psi_x(r)$ is clearly in C^1 and is compactly supported in $[0,2]$. Once again we treat only the two-dimensional case, i.e., $d=2$. Integrating by parts the expression (3.11), we obtain

$$S_N(x) = S(x) + \frac{1}{2\pi}\int_0^2 \left(1-\delta^2 r^2\right)^N \psi_x'(r)\,dr,$$

where

$$\psi_x'(r) = \int_{\partial B_1} \left(\nabla\mathcal{F}^{-1}(h_c)(r\omega).\omega S(x-r\omega) - \mathcal{F}^{-1}(h_c)(r\omega)\nabla S(x-r\omega).\omega\right)d\omega.$$

Therefore

$$\left|S_N(x)-S(x)\right| \leqslant 2M\left(\left\|\frac{1}{\sqrt{r}}\nabla\mathcal{F}^{-1}(h_c)\right\|_{L^2(B_2)} + \left\|\frac{1}{\sqrt{r}}\mathcal{F}^{-1}(h_c)\right\|_{L^2(B_2)}\right)\left(\int_0^1 \left(1-4\delta^2 r^2\right)^{2N} dr\right)^{\frac{1}{2}}.$$

Observe that

$$\nabla\mathcal{F}^{-1}\big(h_c(x)\big)(\xi) = \mathcal{F}^{-1}\big(ixh_c(x)\big)(\xi).$$

Since $xh(x)$ belongs to $C_0^\infty(\mathbb{R}^d)$, the function $\nabla\mathcal{F}^{-1}(xh(x))(\xi)$ is an entire function of 1-exponential type. We deduce from Lemma 3.1 that there exists a positive constant $C>0$, such that

$$\left\|\nabla\mathcal{F}^{-1}\big(xh(x)\big)(z)\right\| \leqslant Ce^{-|z|}, \quad \forall z\in\mathbb{C}^d.$$

A simple change of variables yields to

$$\mathcal{F}^{-1}\big(ixh_c(x)\big)(\xi) = c\mathcal{F}^{-1}\big(ixh(x)\big)(c\xi).$$

Hence

$$\left\|\nabla\mathcal{F}^{-1}(h_c)(\xi)\right\| \leqslant Cce^{-c|\xi|}, \quad \forall\xi\in B_2,$$

and

$$\left\|\frac{1}{\sqrt{r}}\nabla\mathcal{F}^{-1}(h_c)\right\|_{L^2(B_2)} \leqslant (2\pi)^{\frac{1}{2}}\tilde{C},$$

where $\tilde{C}$ is a positive constant independent of c.

Similarly $\mathcal{F}^{-1}(h(x))(\xi)$ is an entire function of 1-exponential type. Since $\mathcal{F}^{-1}(h_c)(\xi) = \mathcal{F}^{-1}(h)(c\xi)$ we have

$$\left|\mathcal{F}^{-1}(h_c)\right\| \leqslant Ce^{-c|\xi|}, \quad \forall\xi\in B_2.$$

Thus

$$\left\| \frac{1}{\sqrt{r}} \mathcal{F}^{-1}(h_c) \right\|_{L^2(B_2)} \leqslant (2\pi)^{\frac{1}{2}} \frac{\tilde{C}}{c}.$$

Consequently

$$\left|S_N(x) - S(x)\right| \leqslant (8\pi)^{\frac{1}{2}} \tilde{C} M \left(1 + \frac{1}{c}\right) \left(\int_0^1 \left(1 - 4\delta^2 r^2\right)^{2N} dr \right)^{\frac{1}{2}}.$$

On the other hand, since $(1 - 4\delta^2 r^2)^N \leqslant e^{-8\delta^2 N r^2}$ for all $r \in [0, 1]$, we arrive at

$$\int_0^1 \left(1 - 4\delta^2 r^2\right)^{2N} dr \leqslant \int_0^1 e^{-8\delta^2 N r^2} dr \leqslant \left(2\pi \int_0^1 e^{-8\delta^2 N r^2} r \, dr \right)^{\frac{1}{2}} = \left(\frac{\pi}{4\delta^2 N} \left(1 - e^{-8\delta^2 N}\right) \right)^{\frac{1}{2}}.$$

Therefore

$$\left|S_N(x) - S(x)\right| \leqslant \left(\frac{4\pi}{\delta}\right)^{\frac{1}{2}} \tilde{C} M \left(1 + \frac{1}{c}\right) \frac{1}{N^{\frac{1}{4}}}, \quad \forall x \in B_1,$$

which completes the proof.

The proof for higher dimension can be conducted again by using the Watson Lemma. □

Lemma 3.6. *Let S be a function in $C^1(\overline{B}_1)$, δ be a constant satisfying $0 < \delta < \frac{1}{2}$ and f is the Fourier transform defined by* (3.2). *If for a positive constant M,*

$$\|S\|_{C^1(\overline{B}_1)} \leqslant M,$$

then

$$\|S\|_{C^0(\overline{B}_1)} \leqslant \frac{C_1}{\delta^{\frac{1}{2}}} M \left(1 + \frac{1}{c}\right) \frac{1}{N^{\frac{1}{4}}} + C_2 \left(1 + \left(\frac{\delta^2}{c^2} N\right)^{\frac{d+1}{2}} \left(\frac{\delta(2N + d - 3)}{ec}\right)^{2N+d-3}\right) \|f\|_{L^1(B_c)},$$

for all $N \geqslant 1$, where C_1 and C_2 are constants that only depend on the dimension d.

Proof. Using the relation $\mathcal{F}^{-1}(h_c)(\xi) = \mathcal{F}^{-1}(h)(c\xi)$ and the fact that $\mathcal{F}^{-1}(h)(\xi)$ is an entire function of 1-exponential type, we have

$$\begin{aligned}
\|h_N\|_{C^0(\overline{B}_c)} &\leqslant C \left(\frac{\delta^2 N}{\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \left|1 - \delta^2 |x|^2\right|^{N-1} e^{-c|x|} dx \\
&\leqslant C \left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}} \int_0^{+\infty} \left|1 - \frac{\delta^2 r^2}{c^2}\right|^{N-1} e^{-r} r^{d-1} dr.
\end{aligned}$$

Furthermore, we deduce from the proof of Lemma 3.4 that

$$\|h_N\|_{C^0(\overline{B}_c)} \leqslant C_2\left(1+\left(\frac{\delta^2}{c^2}N\right)^{\frac{d+1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}\right).$$

Therefore

$$\|S_N\|_{C^0(\overline{B}_1)} \leqslant C_2\left(1+\left(\frac{\delta^2}{c^2}N\right)^{\frac{d+1}{2}}\left(\frac{\delta(2N+d-3)}{ec}\right)^{2N+d-3}\right)\|f\|_{L^1(B_c)}. \tag{3.12}$$

The proof is completed by combining the result of Lemma 3.5 and the estimate (3.12). □

We are ready to prove the second stability theorem for the multi-frequency ISP.

Theorem 3.3. *Let S be a function in $C^1(\overline{B}_1)$ and δ be a constant satisfying $0<\delta<\frac{1}{2}$. Assume that*

$$\epsilon = \|\partial_\nu \psi\|_{L^1(B_c,L^1(\Gamma))} + \|k\psi\|_{L^1(B_c,L^1(\Gamma))} < 1$$

and

$$\|S\|_{C^1(\overline{B}_1)} \leqslant M,$$

where M is a positive constant. The following statements hold:

A) *If $c \geqslant \epsilon^{-4}+1$, then*

$$\|S\|_{C^0(\overline{B}_1)} \leqslant C\left(1+\frac{M}{\delta^{\frac{1}{2}}}\left(1+\frac{1}{c}\right)+\left(\frac{\delta^2}{c}\right)^{\frac{d+1}{2}}\right)\epsilon,$$

where the constant C only depends on the dimension d.

B) *If $c<\epsilon^{-4}+1$ and $\beta\in(0,1)$, then*

$$\|S\|_{C^0(\overline{B}_1)} \leqslant C\left(1+\frac{M}{\delta^{\frac{1}{2}}}\left(1+\frac{1}{c}\right)\left(\frac{\delta^{2\beta}}{c^{2\beta}}+\frac{\delta^{\beta}}{c^{\beta}}\right)^{\frac{1}{4(1+\beta)}}\right)\frac{1}{(\ln(\epsilon^{-1}))^{\frac{1}{4(1+\beta)}}},$$

where C is a constant that only depends on the dimension d and β.

Proof. Let f be the Fourier transform of S, defined by (3.2).

A) Assume that $c \geqslant \|f\|^{-4}_{L^1(B_c)}+1$ and an integer N is chosen such that

$$\|f\|^{-4}_{L^1(B_c)} \leqslant N \leqslant \|f\|^{-4}_{L^1(B_c)}+1.$$

Hence

$$\begin{cases} \dfrac{1}{N^{\frac{1}{4}}} \leqslant \|f\|_{L^1(B_c)}, \\ N\leqslant c. \end{cases}$$

Using the inequalities above and the stability estimate of Lemma 3.4, we get

$$\|S\|_{C^0(\overline{B}_1)} \leqslant \left(\frac{C_1}{\delta^{\frac{1}{2}}} M\left(1+\frac{1}{c}\right) + C_2\left(1+\left(\frac{\delta^2}{c}\right)^{\frac{d+1}{2}}\right)\right)\|f\|_{L^1(B_c)}. \tag{3.13}$$

From (3.1) it follows that

$$\|f\|_{L^1(B_c)} \leqslant \epsilon.$$

By combining (3.13) and the last estimate, the proof of A) is completed.

B) For large N, the stability estimate of Lemma 3.6 can be rewritten as

$$\|S\|_{C^0(\overline{B}_1)} \leqslant \frac{C_1}{\delta^{\frac{1}{2}}} M\left(1+\frac{1}{c}\right)\frac{1}{N^{\frac{1}{4}}} + C_2\|f\|_{L^1(B_c)} + C_2 e^{B_\gamma}\|f\|^{1-\gamma}_{L^1(B_c)},$$

where

$$B_\gamma := \frac{d+1}{2}\ln\left(\frac{\delta^2}{c^2}N\right) + 2N\ln\left(\frac{2\delta N}{ec}\right) - \gamma\ln\left(\|f\|^{-1}_{L^1(B_c)}\right).$$

Let β be a fixed real in $(0,1)$ and set $N = (\lambda^{-1}\ln(\|f\|^{-\gamma}_{L^1(B_c)}))^{\frac{1}{1+\beta}}$, where λ is a positive constant. Moreover, one may choose λ so that $B_\gamma \leqslant 0$, which implies

$$\|S\|_{C^0(\overline{B}_1)} \leqslant \frac{C_1}{\delta^{\frac{1}{2}}} M\left(1+\frac{1}{c}\right)\frac{1}{N^{\frac{1}{4}}} + 2C_2\|f\|^{1-\gamma}_{L^1(B_c)}.$$

Using the fact that

$$\ln(t) \leqslant \frac{t^\beta}{e\beta}, \quad \forall t \geqslant 1,$$

we get

$$B_\gamma \leqslant \left(\frac{\lambda^{-1}}{e\beta}\left(\frac{d+1}{2}\left(\frac{\delta}{c}\right)^{2\beta} + 2\left(\frac{2\delta}{ec}\right)^{\beta}\right) - 1\right)\ln\left(\|f\|^{-\gamma}_{L^1(B_c)}\right).$$

By taking $\lambda = \lambda_c := \frac{1}{e\beta}(\frac{d+1}{2}(\frac{\delta}{c})^{2\beta} + 2(\frac{2\delta}{ec})^{\beta})$, $\|f\|_{L^1(B_c)} < 1$, and $\gamma = \frac{1}{2}$, we obtain

$$\|S\|_{C^0(\overline{B}_1)} \leqslant \left(2^{\frac{1}{4(1+\beta)}}\frac{C_1}{\delta^{\frac{1}{2}}} M\left(1+\frac{1}{c}\right)\lambda_c^{\frac{1}{4(1+\beta)}} + C_2\right)\frac{1}{(\ln(\|f\|^{-1}_{L^1(B_c)}))^{\frac{1}{4(1+\beta)}}}. \tag{3.14}$$

By noting that

$$\|f\|_{L^1(B_c)} \leqslant \epsilon,$$

the proof is completed by combining the above estimate and (3.14). □

The following corollary corresponds to the results of Theorem 3.3 with $\delta = \frac{1}{4}$ and $\beta = \frac{1}{2}$.

Corollary 3.2. *Let S be a function in $C^1(\overline{B}_1)$. Assume that*

$$\epsilon = \|\partial_\nu \psi\|_{L^1(B_c, L^1(\Gamma))} + \|k\psi\|_{L^1(B_c, L^1(\Gamma))} < 1$$

and

$$\|S\|_{C^1(\overline{B}_1)} \leqslant M,$$

where M is a positive constant. The following statements hold:

A) *If $c \geqslant \epsilon^{-4} + 1$, then*

$$\|S\|_{C^0(\overline{B}_1)} \leqslant C\left(1 + 2M\left(1 + \frac{1}{c}\right) + \left(\frac{1}{16c}\right)^{\frac{d+1}{2}}\right)\epsilon,$$

where the constant C only depends on the dimension d.

B) *If $c < \epsilon^{-4} + 1$, then*

$$\|S\|_{C^0(\overline{B}_1)} \leqslant C\left(1 + 2M\left(1 + \frac{1}{c}\right)\left(\frac{1}{4c} + \frac{1}{2\sqrt{c}}\right)^{\frac{1}{6}}\right)\frac{1}{(\ln(\epsilon^{-1}))^{\frac{1}{6}}},$$

where C is a constant that only depends on the dimension d.

We deduce from Theorem 3.3 and Corollary 3.2 that if the source function S is compactly supported in B_1 and the measurement $\psi(k, x)|_\Gamma$ is taken for $k \in [0, c]$ with a noise ϵ, then the recovery of S is linearly and logarithmic stable with respect to the noise for large and small c, respectively.

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Appendix A

Here, we present some useful properties of the function $h_N(x)$.

Let $\varphi(r,\theta) = f(r)e^{il\theta}$, where (r,θ) is the polar coordinate. Further, the Fourier transform of the function φ may be computed:

$$\mathcal{F}\big(f(\rho)e^{il\beta}\big)(\xi) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} f(\rho)e^{il\beta}e^{-i(\xi_1\rho\cos(\beta)+\xi_2\rho\sin(\beta))}\rho\,d\rho\,d\beta,$$

where (ρ,β) denotes the polar coordinate of $\xi = (\xi_1, \xi_2)$. Define $\lambda = \beta - \theta$, the angle between ξ and (ρ,β) and compute as follows

$$\mathcal{F}\big(f(\rho)e^{il\beta}\big)(r,\theta) = \frac{1}{2\pi}\int_0^\infty f(\rho)\int_0^{2\pi} e^{il(\lambda+\theta)}e^{-ir\rho\cos(\lambda)}\,d\lambda\,\rho\,d\rho$$
$$= \frac{1}{2\pi}e^{il(\theta+\pi)}\int_0^\infty f(\rho)\rho\,d\rho\int_{-\pi}^{\pi} e^{il\lambda+ir\rho\cos(\lambda)}\,d\lambda.$$

It is well known, see for instance [1], that

$$\int_{-\pi}^{\pi} e^{il\lambda+ir\rho\cos(\lambda)}\,d\lambda = 2\pi\, J_l(r\rho)e^{il\frac{\pi}{2}}.$$

Hence

$$\mathcal{F}(\varphi)(r,\theta) = e^{il(\theta+\frac{3\pi}{2})}\int_0^\infty f(\rho)\,J_l(r\rho)\rho\,d\rho.$$

Recall the expression of $h_N(x) = \mathcal{F}(\delta_N)(x)$, where δ_N is the product of a polynomial with the Fourier inverse of $h(x)$:

$$\delta_N(x) = \left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}}\left(1-\frac{\delta^2|x|^2}{c^2}\right)^{N-1}\mathcal{F}^{-1}(h)(x).$$

Since the function $\delta_N(x) = \delta_N(|x|)$ is radially symmetric, its Fourier inverse is also radially symmetric.

Introduce the multi-index $\alpha = (\alpha_1, \alpha_d, \dots, \alpha_d)$ in $\mathbb{N}^d$. Let

$$\partial_\alpha h(x) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\dots\partial_{x_d}^{\alpha_d}h(x)$$

be the α partial derivative of $h(x)$. We have $\mathcal{F}^{-1}(\partial^\alpha h)(x) = i^{|\alpha|}x^\alpha\mathcal{F}^{-1}(h)(x)$. A straightforward calculation yields

$$\delta_N(\xi) = \left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}}\sum_{j=0}^{N-1}\frac{(N-1)!}{j!(N-j-1)!}\left(-\frac{\delta^2}{c^2}\right)^j|\xi|^{2j}\mathcal{F}^{-1}(h)(\xi)$$
$$= \mathcal{F}^{-1}\left(\left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}}\sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{(N-1)!}{j!(N-j-1)!}\left(\frac{\delta^2}{c^2}\right)^j\frac{j!}{\alpha_1!\alpha_2!}\partial_{2\alpha}h(.)\right)(\xi).$$

Therefore

$$h_N(x) = \left(\frac{\delta^2 N}{\pi c^2}\right)^{\frac{d}{2}}\sum_{j=0}^{N-1}\sum_{|\alpha|=j}\frac{(N-1)!}{j!(N-j-1)!}\left(\frac{\delta^2}{c^2}\right)^j\frac{j!}{\alpha_1!\alpha_2!}\partial_{2\alpha}h(x).$$

Consequently $h_N(x)$ is compactly supported in B_1.

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