

SCATTERING BY A PERIODIC ARRAY OF SUBWAVELENGTH SLITS I: FIELD ENHANCEMENT IN THE DIFFRACTION REGIME*

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Abstract. This is the first part in a series of two papers that are concerned with the quantitative analysis of the electromagnetic field enhancement and anomalous diffraction by a periodic array of perfect conducting (PEC) subwavelength slits. The scattering problem in the diffraction regime is investigated in this part, for which the size of the period is comparable to the incident wavelength. We distinguish scattering resonances and real eigenvalues, and derive their asymptotic expansions when they are away from the Rayleigh cutoff frequencies. Furthermore, we present quantitative analysis of the field enhancement at resonant frequencies, by quantifying both the enhancement order and the associated resonant modes. The field enhancement near the Rayleigh cutoff frequencies is also investigated. It is demonstrated that the field enhancement at resonant frequencies becomes weaker if those frequencies are close to one of the Rayleigh cutoff frequencies. Finally, we also characterize the embedded eigenvalues for the underlying periodic structure, and point out that transmission anomalies such as the Fano resonant phenomenon do not occur for the PEC narrow slit array.

Key words. electromagnetic field enhancement, nano gap, subwavelength structure, grating, Helmholtz equation

AMS subject classifications. 35C20, 35Q60, 35P30

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1. Introduction. There has been increasing interest in the electromagnetic scattering by subwavelength apertures or holes in recent years, due to their significant applications in biological and chemical sensing, near-field spectroscopy, etc. [1, 10, 16, 17, 18, 20, 21, 22, 27, 30, 31, 33]. Such subwavelength structures can generate extraordinary optical transmission and strongly enhanced local electromagnetic fields, which lead to extremely high sensitivity in the sensing or imaging of biological or chemical samples. However, as of today there are still controversies over the mechanisms contributing to the anomalous field enhancement [18]. The complication has to do with the multiscale nature of the underlying metallic structures as well as various enhancement behaviors that it induces. For instance, the enhancement can be attributed to surface plasmonic resonance [17, 18], nonplasmonic resonance [31, 33], or even without the resonant effect (cf. [21, 22, 23]).

Very recently, for a single narrow slit perforated in a perfectly conducting slab, we have presented quantitative analysis of the field enhancement, which provides a complete picture for its enhancement mechanisms [23]. We also refer to [14, 15] a closely related problem of scattering by subwavelength cavities. In this paper and its sequel [24], we investigate the scattering and field enhancement when the slab is patterned with a periodic array of narrow slits. The physics becomes richer compared to the single slit case. In addition, the quantitative studies of field enhancement

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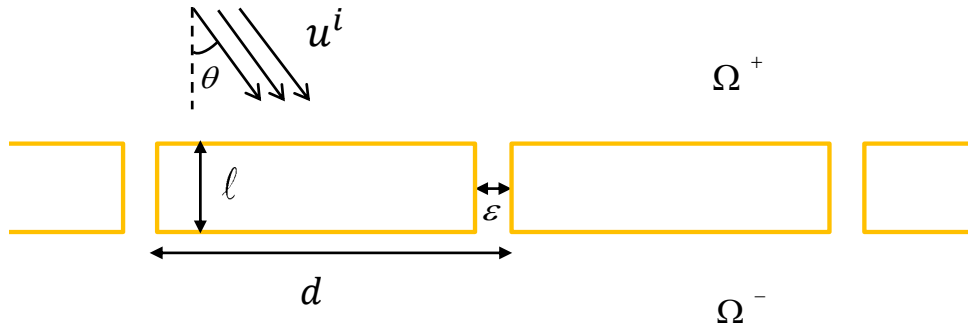


FIG. 1. Geometry of the scattering problem. The slits S_ε are arranged periodically with the period d , and each slit has a rectangular shape of length ℓ and width ε , respectively. The scaling of geometrical parameters is given by $\varepsilon \ll \ell \sim d \sim \lambda$. The domains above and below the perfect conductor slab are denoted as Ω^+ and Ω^- , respectively, and the domain exterior to the perfect conductor is denoted as Ω_ε , which consists of S_ε , Ω^+ , and Ω^- .

also present new mathematical challenges: (i) while the scattering problem for the single slit attains a unique solution, there exists a set of singular frequencies (real eigenvalues) for the periodic structure; (ii) in contrast to the single slit case, the field enhancement factor for the periodic case would depend on the size of the period and the incident angle for a given frequency; (iii) the scattering by the periodic structure will exhibit so-called Rayleigh anomaly when certain propagating spatial harmonics turn into an evanescent mode, which gives rise to diffractive anomaly near such cutoff frequencies; (iv) the field enhancement mechanisms in the diffraction regime and the homogenization regime are different and need separate investigations. The incident wavelength is comparable to the size of the period in the former, but much larger in the latter. As is to be demonstrated in this paper and in [24], while the field enhancement in the diffraction regime is mostly attributed to the resonance effect, the enhancement and diffraction anomaly in the homogenization regimes can be induced by nonresonance effects.

We now present the setup of our scattering problem. We consider a perfectly conducting slab that is perforated with a periodic array of slits and the geometry of its cross section is depicted in Figure 1. The slab occupies the domain $\{(x_1, x_2) \mid 0 < x_2 < \ell\}$ on the x_1x_2 plane, where ℓ is the thickness of the metal. The slits, which are invariant along the x_3 direction, occupy the region $S_\varepsilon = \bigcup_{n=0}^{\infty} (S_\varepsilon^{(0)} + nd)$, where d is the size of the period, and $S_\varepsilon^{(0)} := \{(x_1, x_2) \mid 0 < x_1 < \varepsilon, 0 < x_2 < \ell\}$. Let us denote the semi-infinite domains above and below the slab by Ω^+ and Ω^- , respectively. We also denote by Ω_ε the domain exterior to the perfect conductor, i.e., $\Omega_\varepsilon = \Omega^+ \cup \Omega^- \cup S_\varepsilon$, and ν the unit outward normal pointing to the exterior domain Ω^+ or Ω^- .

In this paper, we are interested in the case when the width of the slit ε is much smaller than the thickness of the slab ℓ , and the size of the period d is comparable to the thickness of the slab ℓ . Furthermore, the incident wavelength λ is comparable to d such that the problem under consideration is in the diffraction regime. In sum, we assume that $\varepsilon \ll \ell$ and $d \sim \ell \sim \lambda$ for the parameters in the scattering problem under consideration. For clarity, we shall assume $\ell = 1$ in all technical derivations throughout the paper. The enhancement theory for the case of $\ell \neq 1$ then follows by a normalization process and a scaling argument.

Assume that a polarized time-harmonic electromagnetic wave impinges upon the perfect conductor from above. We consider the transverse magnetic case where the incident magnetic field is perpendicular to the x_1x_2 plane, and its x_3 component is

given by the scalar function $u^i = e^{ik(\sin\theta x_1 - \cos\theta(x_2-1))}$. Here k is the wavenumber and $\theta \in (-\pi/2, \pi/2)$ is the incident angle. The total field u_ε , which consists of the incident wave u^i and the scattered field u_ε^s in Ω^+ and the scattered field u_ε^s only in Ω^- , satisfies the Helmholtz equation

$$(1.1) \quad \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon,$$

and the boundary condition

$$(1.2) \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Let $\kappa = k \sin\theta$, we look for quasi-periodic solutions such that $u_\varepsilon(x_1, x_2) = e^{i\kappa x_1} \tilde{u}_\varepsilon(x_1, x_2)$, where \tilde{u}_ε is a periodic function with $\tilde{u}_\varepsilon(x_1 + d, x_2) = \tilde{u}_\varepsilon(x_1, x_2)$ or, equivalently,

$$(1.3) \quad u_\varepsilon(x_1 + d, x_2) = e^{i\kappa d} u_\varepsilon(x_1, x_2).$$

Define

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \sqrt{k^2 - \kappa_n^2},$$

where the function $f(z) = \sqrt{z}$ is understood as an analytic function defined in the domain $\mathbf{C} \setminus \{-it : t \geq 0\}$ by

$$z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{\frac{1}{2}i \arg z}$$

throughout the paper. Then it can be shown that the outgoing scattered field adopts the following Rayleigh–Bloch expansion (cf. [11, 13, 29])

$$(1.4) \quad u_\varepsilon^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} u_n^{s,+} e^{i\kappa_n x_1 + i\zeta_n x_2} \quad \text{and} \quad u_\varepsilon^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} u_n^{s,-} e^{i\kappa_n x_1 - i\zeta_n x_2}$$

in the domain Ω^+ and Ω^- , respectively, for some coefficients $u_n^{s,\pm}$. The expansions (1.4) are usually referred to as the outgoing radiation condition, and are imposed for the scattered field above and below the metallic slab.

Due to the quasi-periodicity of the solution, one can restrict κ to the first Brillouin zone $(-\pi/d, \pi/d]$. Such a κ is called the reduced wave vector component [13, 29]. For given k, κ, d , we denote three sets of indices: $I_1 = I_1(k, \kappa, d) = \{n : |\kappa + 2\pi n/d| < k\}$, $I_2 = I_2(k, \kappa, d) = \{n : |\kappa + 2\pi n/d| > k\}$, and $I_3 = I_3(k, \kappa, d) = \{n : |\kappa + 2\pi n/d| = k\}$. The spacial harmonics $e^{i\kappa_n x_1 + i\zeta_n x_2}$ are called propagating modes, evanescent modes, and linear modes for n belonging to I_1 , I_2 , and I_3 , respectively.

In contrast to the scattering problem for a single slit where one has uniqueness, the solution to the scattering problem (1.1)–(1.4) may not be unique. Indeed, the corresponding homogeneous problem with $u^i = 0$ may attain nontrivial solutions (k, u_ε) for $k \in \mathbf{R}$ [13, 28, 32]. Such a real-valued k is called a real eigenvalue or singular frequency and u_ε is the corresponding guided mode or surface bound state that decays exponentially away from the periodic structure [13, 28, 32]. On the other hand, there may also exist complex-valued k 's such that the homogeneous problem attains nontrivial solutions. Such k 's are called resonances (or scattering resonances) of the scattering problem, and the associated nontrivial solutions are called leaky modes (or quasi-normal modes). If the frequency of the incident wave is close to the real part of the resonance (resonance frequency), then an enhancement of scattering is expected

if the imaginary part of the resonance is small. This is the mechanism of resonant scattering. We refer to [3, 4, 5, 6, 7, 8, 9] for recent mathematical investigation of the interesting applications of resonances in superresolution/superfocusing and metasurfaces. Finally, if $I_3(k, \kappa) \neq \emptyset$, then near a cutoff frequency, where $k = \pm(\kappa + 2\pi n/d)$ for some integer n , the propagating spatial harmonic mode $e^{i\kappa_n x_1 \pm i\zeta_n x_2}$ becomes an evanescent mode or vice versa. As such the diffracted field will exhibit anomalous behaviors and this is the so-called Rayleigh anomaly [26]. If resonances occur near such cutoff frequencies, then the enhancement behavior at resonant frequencies will be different from that of resonant phenomena away from the cutoff frequencies.

In this paper, by using layer potential techniques, variational approaches, and asymptotic analysis, we explore the field enhancement mechanism for the scattering problem in the diffraction regime. We derive the asymptotic expansions for both real eigenvalues and complex scattering resonances, which lie below and above the light line $k = |\kappa|$, respectively. It is known that surface bound states, for which the relation $k < |\kappa|$ holds, do not couple with a plane incident wave $u^i = e^{i(\kappa x_1 - \zeta(x_2 - 1))}$ that satisfies $\kappa = k \sin \theta$, whereas the quasi-modes, for which $|\kappa| < k$ holds, can be excited if the incident frequency coincides with one of the resonant frequencies. We analyze quantitatively the field enhancement at resonant frequencies, and show that enhancement of order $O(\varepsilon^{-1})$ is achieved. The enhanced wave modes are also characterized in both near-field and far-field zones. In addition, we demonstrate that if the resonances are near to one of the Rayleigh cutoff frequencies with a distance $O(\varepsilon^{2\tau})$, where $0 < \tau < 1$, then the field enhancement becomes weaker and is of order $O(\varepsilon^{\tau-1})$.

The rest of the paper is organized as follows. We introduce layer potentials for the scattering problem and the boundary integral formulations in section 2. The asymptotic expansion for the solution to the scattering problem is derived in section 3, which lays the foundation for the quantitative analysis of field enhancement. The asymptotic expansions of real eigenvalues and complex resonances, and the investigation of field enhancement at resonant frequencies are presented in section 4. In section 5, we study the field enhancement when the resonant frequency is close to the Rayleigh cutoff frequencies. Finally, we briefly discuss the embedded eigenvalues associated with the scattering problem.

2. Boundary integral formulation. For each fixed $\kappa \in (-\pi/d, \pi/d]$, let $g^d(x, y) = g^d(x, y; \kappa)$ be the quasi-periodic Green's function solving the following equation:

$$\Delta g^d(x, y) + k^2 g^d(x, y) = e^{i\kappa(x_1 - y_1)} \sum_{n=-\infty}^{\infty} \delta(x_1 - y_1 - nd) \delta(x_2 - y_2) \quad x, y \in \mathbf{R}^2.$$

Then (cf. [25])

$$(2.1) \quad g^d(x, y; \kappa) = -\frac{i}{2d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n(x_1 - y_1) + i\zeta_n(k)|x_2 - y_2|},$$

where

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \begin{cases} \sqrt{k^2 - \kappa_n^2}, & |\kappa_n| < k, \\ i\sqrt{\kappa_n^2 - k^2}, & |\kappa_n| > k. \end{cases}$$

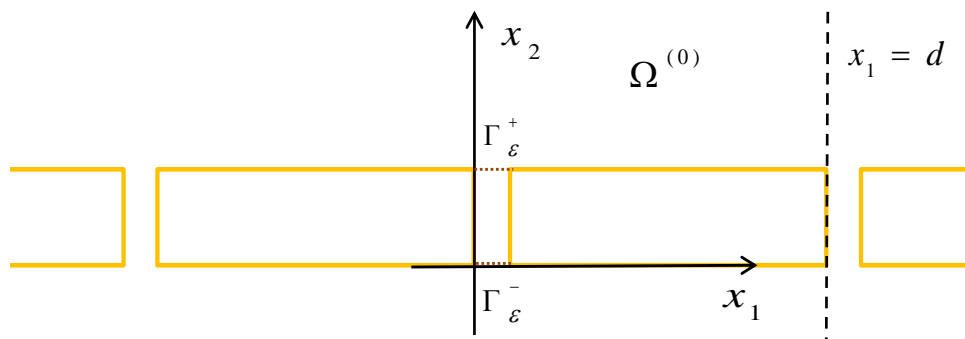


FIG. 2. Problem geometry in one reference period $\Omega^{(0)}$.

We define the exterior Green's function with the Neumann boundary condition as $g^e(x, y) = g^e(x, y; \kappa) = g^d(x, y; \kappa) + g^d(x', y, \kappa)$, where

$$x' = \begin{cases} (x_1, 2 - x_2) & \text{if } x, y \in \Omega^+, \\ (x_1, -x_2) & \text{if } x, y \in \Omega^-. \end{cases}$$

It is easy to verify that $\frac{\partial g^e(x, y; \kappa)}{\partial \nu_y} = 0$ on $\{y_2 = 1\}$ and $\{y_2 = 0\}$.

We also define the Green's function $g_\varepsilon^i(x, y)$ in the slit $S_\varepsilon^{(0)}$ with the Neumann boundary condition as

$$g_\varepsilon^i(x, y) = \sum_{m, n=0}^{\infty} c_{mn} \phi_{mn}(x) \phi_{mn}(y).$$

Here $c_{mn} = \frac{1}{k^2 - (m\pi/\varepsilon)^2 - (n\pi)^2}$, $\phi_{mn} = \sqrt{\frac{a_{mn}}{\varepsilon}} \cos(\frac{m\pi x_1}{\varepsilon}) \cos(n\pi x_2)$, and the coefficient a_{mn} is given by

$$a_{mn} = \begin{cases} 1, & m = n = 0, \\ 2, & m = 0, n \geq 1 \quad \text{or} \quad n = 0, m \geq 1, \\ 4, & m \geq 1, n \geq 1. \end{cases}$$

It is easy to check that

$$\Delta g_\varepsilon^i(x, y) + k^2 g_\varepsilon^i(x, y) = \delta(x - y), \quad x, y \in S_\varepsilon^{(0)}.$$

To formulate the scattering problem as boundary integral equations, let us reduce the problem to one reference cell $\Omega^{(0)} := \{x \in \mathbf{R}^2 \mid 0 < x_1 < d\}$ as shown in Figure 2. Recall that the slit sitting in $\Omega^{(0)}$ is given by $S_\varepsilon^{(0)} := \{(x_1, x_2) \mid 0 < vx_1 < \varepsilon, 0 < x_2 < \ell\}$. We denote the upper and lower aperture of the slit by Γ_ε^+ and Γ_ε^- , respectively, (see Figure 2).

LEMMA 2.1. *Let $u_\varepsilon(x)$ be the solution of the scattering problem (1.1)–(1.4), then*

$$\begin{aligned} u_\varepsilon(x) &= \int_{\Gamma_\varepsilon^+} g^e(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y + u^i + u^r \quad \text{for } x \in \Omega^{(0)} \cap \Omega^+, \\ u_\varepsilon(x) &= - \int_{\Gamma_\varepsilon^-} g^e(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y \quad \text{for } x \in \Omega^{(0)} \cap \Omega^-, \\ u_\varepsilon(x) &= \int_{\Gamma_\varepsilon^-} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y - \int_{\Gamma_\varepsilon^+} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y \quad \text{for } x \in S_\varepsilon^{(0)}. \end{aligned}$$

Here $u^r = e^{i(\kappa x_1 + \zeta(x_2 - 1))}$ is the reflected field of the perfect conducting ground plane $\{x_2 = 1\}$.

Proof. For $x, y \in \Omega^{(0)} \cap \Omega^+$, define $\tilde{g}^e(x, y) = e^{-i\kappa(x_1 - y_1)}g^e(x, y)$ and $\tilde{u}_\varepsilon^s(y) = e^{-i\kappa y_1}u_\varepsilon^s(y)$. Then $\tilde{g}^e(x, y)$ and $\tilde{u}_\varepsilon^s(y)$ are periodic with respect to y_1 . Furthermore, a direction calculation yields

$$\Delta \tilde{g}^e(x, y) - i2\kappa \frac{\partial \tilde{g}^e(x, y)}{\partial y_1} + (k^2 - \kappa^2)\tilde{g}^e(x, y) = \sum_{n=-\infty}^{\infty} \delta(x_1 - y_1 - nd)\delta(x_2 - y_2),$$

$$\Delta \tilde{u}_\varepsilon^s(y) + i2\kappa \frac{\partial \tilde{u}_\varepsilon^s(y)}{\partial y_1} + (k^2 - \kappa^2)\tilde{u}_\varepsilon^s(y) = 0.$$

Choose $R > x$. Define the bounded domain above the perfect conductor by

$$\Omega_R^+ := \{x \in \Omega^{(0)} \mid 1 < x_2 < R\}.$$

Denote the boundary of Ω_R^+ by $\partial\Omega_R^+$. Let $\Gamma_R^+ := \{(x_1, R) \mid 0 < x_1 < d\}$ and $\Gamma_1^+ := \{(x_1, 1) \mid 0 < x_1 < d\}$ be its upper and lower boundaries, respectively. An application of the Green's second identity and the divergence theorem over the domain Ω_R^+ yield

$$\begin{aligned} \tilde{u}_\varepsilon^s(x) &= \int_{\Omega_R^+} \Delta \tilde{g}^e(x, y)\tilde{u}_\varepsilon^s(y) - \Delta \tilde{u}_\varepsilon^s(y)\tilde{g}^e(x, y) - i2\kappa \int_{\Omega_R^+} \frac{\partial}{\partial y_1} (\tilde{g}^e(x, y)\tilde{u}_\varepsilon^s(y)) dy \\ &= \int_{\partial\Omega_R^+} \frac{\partial \tilde{g}^e(x, y)}{\partial \nu_y} \tilde{u}_\varepsilon^s(y) - \tilde{g}^e(x, y) \frac{\partial \tilde{u}_\varepsilon^s(y)}{\partial \nu_y} ds_y - i2\kappa \int_{\partial\Omega_R^+} (\tilde{g}^e(x, y)\tilde{u}_\varepsilon^s(y))\nu_1 ds_y. \end{aligned}$$

Using the fact that $\tilde{g}^e(x, y)$ and $\tilde{u}_\varepsilon^s(y)$ are periodic along y_1 , the second integral vanishes by noting that $\nu_1 = 0$ on the horizontal boundaries and ν takes opposite signs on two vertical boundaries. Substituting the Rayleigh expansions (1.4) and (2.1), it follows that

$$\int_{\Gamma_R^+} \frac{\partial \tilde{g}^e(x, y)}{\partial \nu_y} \tilde{u}_\varepsilon^s(y) - \tilde{g}^e(x, y) \frac{\partial \tilde{u}_\varepsilon^s(y)}{\partial \nu_y} ds_y = 0.$$

Thus, again applying the periodic boundary conditions for $\tilde{g}^e(x, y)$ and $\tilde{u}_\varepsilon^s(y)$, we have

$$\tilde{u}_\varepsilon^s(x) = \int_{\Gamma_1^+} \tilde{g}^e(x, y) \frac{\partial \tilde{u}_\varepsilon^s(y)}{\partial y_2} ds_y$$

or, equivalently,

$$\begin{aligned} \tilde{u}_\varepsilon(x) &= \int_{\Gamma_1^+} \tilde{g}^e(x, y) \left(\frac{\partial \tilde{u}_\varepsilon(y)}{\partial y_2} - \frac{\partial \tilde{u}^i(y)}{\partial y_2} \right) ds_y + \tilde{u}^i(x) \\ &= \int_{\Gamma_1^+} \tilde{g}^e(x, y) \frac{\partial \tilde{u}_\varepsilon(y)}{\partial y_2} ds_y - \int_{\Gamma_1^+} \tilde{g}^e(x, y) \frac{\partial \tilde{u}^i(y)}{\partial y_2} ds_y + \tilde{u}^i(x), \end{aligned}$$

where $\tilde{u}^i(x) = e^{-i\kappa x_1}u^i(x)$. A direct calculation leads to

$$\int_{\Gamma_1^+} \tilde{g}^e(x, y) \frac{\partial \tilde{u}^i(y)}{\partial y_2} ds_y = -e^{i\zeta(x_2 - 1)}.$$

Since $g^e(x, y) = e^{i\kappa(x_1 - y_1)}\tilde{g}^e(x, y)$, $u_\varepsilon(y) = e^{i\kappa y_1}\tilde{u}_\varepsilon(y)$ and $u^i(x) = e^{i\kappa x_1}\tilde{u}^i(x)$. It follows that

$$u_\varepsilon(x) = \int_{\Gamma_\varepsilon^+} g^e(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y + u^i(x) + u^r(x),$$

where we have used the boundary condition $\partial_\nu u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$.

Following similar lines to the above, it can be shown that

$$u_\varepsilon(x) = - \int_{\Gamma_\varepsilon^-} g^\varepsilon(x, y) \frac{\partial u_\varepsilon}{\partial y_2} ds_y \quad \text{for } x \in \Omega^{(0)} \cap \Omega^-.$$

On the other hand, an application of Green's second identity over the slit $S_\varepsilon^{(0)}$ gives rise to

$$u_\varepsilon(x) = - \int_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon}{\partial \nu_y} ds_y \quad \text{for } x \in S_\varepsilon^{(0)}.$$

The lemma is proved. \square

Based upon Lemma 2.1 and the continuity of the single layer potential (cf. [19]), we obtain the following boundary integral equations defined over the slit apertures Γ_ε^\pm .

LEMMA 2.2. *The following hold for the solution to the scattering problem (1.1)–(1.4):*

$$(2.2) \quad u_\varepsilon(x) = \int_{\Gamma_\varepsilon^+} g^\varepsilon(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y + u^i + u^r \quad \text{for } x \in \Gamma_\varepsilon^+,$$

$$(2.3) \quad u_\varepsilon(x) = - \int_{\Gamma_\varepsilon^-} g^\varepsilon(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y \quad \text{for } x \in \Gamma_\varepsilon^-,$$

$$(2.4) \quad u_\varepsilon(x) = \int_{\Gamma_\varepsilon^-} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y - \int_{\Gamma_\varepsilon^+} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y \quad \text{for } x \in \Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-.$$

An application of the above lemma leads to the following system of integral equations:

$$(2.5) \quad \left\{ \begin{array}{l} \int_{\Gamma_\varepsilon^+} g^\varepsilon(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y + \int_{\Gamma_\varepsilon^+} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y \\ \quad - \int_{\Gamma_\varepsilon^-} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y + u^i + u^r = 0 \quad \text{on } \Gamma_\varepsilon^+, \\ \int_{\Gamma_\varepsilon^-} g^\varepsilon(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y + \int_{\Gamma_\varepsilon^+} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y \\ \quad - \int_{\Gamma_\varepsilon^-} g_\varepsilon^i(x, y) \frac{\partial u_\varepsilon(y)}{\partial y_2} ds_y = 0 \quad \text{on } \Gamma_\varepsilon^-. \end{array} \right.$$

PROPOSITION 2.3. *The scattering problem (1.1)–(1.4) is equivalent to the system of boundary integral equations (2.5).*

It is clear that

$$\frac{\partial u_\varepsilon}{\partial \nu} \Big|_{\Gamma_\varepsilon^+} = \frac{\partial u_\varepsilon}{\partial y_2}(y_1, 1), \quad \frac{\partial u_\varepsilon}{\partial \nu} \Big|_{\Gamma_\varepsilon^-} = - \frac{\partial u_\varepsilon}{\partial y_2}(y_1, 0), \quad (u^i + u^r)|_{\Gamma_\varepsilon^+} = 2e^{i\kappa x_1}.$$

Note that the above functions are defined over narrow intervals with size $\varepsilon \ll 1$. To facilitate the analysis, we let $X = x_1/\varepsilon$ and $Y = y_1/\varepsilon$ define the following rescaled

quantities:

$$\begin{aligned} \varphi_1(Y) &:= -\frac{\partial u_\varepsilon}{\partial y_2}(\varepsilon Y, 1), \\ \varphi_2(Y) &:= \frac{\partial u_\varepsilon}{\partial y_2}(\varepsilon Y, 0), \\ f(X) &:= (u^i + u^r)(\varepsilon X, 1) = 2e^{i\kappa\varepsilon X}, \\ G_\varepsilon^e(X, Y) &= G_\varepsilon^e(X, Y, \kappa) := g^e(\varepsilon X, 1; \varepsilon Y, 1) = g^e(\varepsilon X, 0; \varepsilon Y, 0) \\ &= -\frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)}, \\ G_\varepsilon^i(X, Y) &:= g_\varepsilon^i(\varepsilon X, 1; \varepsilon Y, 1) = g_\varepsilon^i(\varepsilon X, 0; \varepsilon Y, 0) \\ &= \sum_{m,n=0}^{\infty} \frac{c_{mn} a_{mn}}{\varepsilon} \cos(m\pi X) \cos(m\pi Y), \\ \tilde{G}_\varepsilon^i(X, Y) &:= g_\varepsilon^i(\varepsilon X, 1; \varepsilon Y, 0) = g_\varepsilon^i(\varepsilon X, 0; \varepsilon Y, 1) \\ &= \sum_{m,n=0}^{\infty} \frac{(-1)^n c_{mn} a_{mn}}{\varepsilon} \cos(m\pi X) \cos(m\pi Y). \end{aligned}$$

We also define three boundary integral operators:

$$(2.6) \quad (T^e \varphi)(X) = \int_0^1 G_\varepsilon^e(X, Y) \varphi(Y) dY, \quad X \in (0, 1),$$

$$(2.7) \quad (T^i \varphi)(X) = \int_0^1 G_\varepsilon^i(X, Y) \varphi(Y) dY, \quad X \in (0, 1),$$

$$(2.8) \quad (\tilde{T}^i \varphi)(X) = \int_0^1 \tilde{G}_\varepsilon^i(X, Y) \varphi(Y) dY, \quad X \in (0, 1).$$

By a change of variable $x_1 = \varepsilon X$ and $y_1 = \varepsilon Y$ in (2.5), the following proposition follows.

PROPOSITION 2.4. *The system of (2.5) is equivalent to the following one:*

$$(2.9) \quad \begin{bmatrix} T^e + T^i & \tilde{T}^i \\ \tilde{T}^i & T^e + T^i \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} f/\varepsilon \\ 0 \end{bmatrix}.$$

3. Solution to the scattering problem. In this section, based upon the integral equation formulation, we derive the asymptotic expansion of the solution to the scattering problem (1.1)–(1.4). From the derived expansion formulas, we obtain and classify different conditions for extraordinary field enhancement for the underlying periodic structure. In addition, the expansions also lead to explicit asymptotic formulas for the enhanced wavefields presented in sections 4 and 5.

We begin with asymptotic expansions of the boundary integral operators T^e , T^i , and \tilde{T}^i . Then the solution for the integral equation system (2.9) will be obtained, followed by the asymptotic expansion of the waves in the far-field and near-field zones.

3.1. Preliminaries. We introduce several function spaces to be used in the rest of the paper. Let $s \in \mathbf{R}$, we denote by $H^s(\mathbf{R})$ the standard fractional Sobolev space with the norm

$$\|u\|_{H^s(\mathbf{R})}^2 = \int_{\mathbf{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where \hat{u} is the Fourier transform of u . Let I be a bounded open interval in \mathbf{R} and define

$$H^s(I) := \{u = U|_I \mid U \in H^s(\mathbf{R})\}.$$

Then $H^s(I)$ is a Hilbert space with the norm

$$\|u\|_{H^s(I)} = \inf\{\|U\|_{H^s(\mathbf{R})} \mid U \in H^s(\mathbf{R}) \text{ and } U|_I = u\}.$$

We also define

$$\tilde{H}^s(I) := \{u = U|_I \mid U \in H^s(\mathbf{R}) \text{ and } \text{supp } U \subset \bar{I}\}.$$

One can show that the space $\tilde{H}^s(I)$ is the dual of $H^{-s}(I)$ and the norm for $\tilde{H}^s(I)$ can be defined via the duality [2]. As such $\tilde{H}^s(I)$ is also a Hilbert space. Here and henceforth, for simplicity, we denote

$$V_1 = \tilde{H}^{-\frac{1}{2}}(0, 1) \quad \text{and} \quad V_2 = H^{\frac{1}{2}}(0, 1).$$

The duality between V_1 and V_2 will be denoted by $\langle u, v \rangle$ for any $u \in V_1, v \in V_2$.

3.2. Asymptotic expansion of the boundary integral operators. For each fixed $\kappa \in (-\pi/d, \pi/d]$, we derive the asymptotic expansion of the boundary integral operators T^e, T^i , and \tilde{T}^i . This is performed for k away from the Rayleigh cut-off frequencies, where $k = \kappa + 2\pi n/d$ for some integer n . In this way we exclude the scenario where the Green's function $G_\varepsilon^e(X, Y)$ is not well defined because of a vanishing $\zeta_n(k)$. To this end, we introduce a parameter δ such that

$$(3.1) \quad \delta = O(\varepsilon^{2\tau})$$

for some constant τ with $0 \leq \tau < 1$.

Let $B_\delta(z)$ be the disk with radius δ centered at z . Here and henceforth, we define

$$B_{\kappa, \delta} := \bigcup_{n=-\infty}^{\infty} B_\delta(\kappa + 2\pi n/d),$$

and restrict k to the domain $\mathbf{R}^+ \setminus B_{\kappa, \delta}$. The following notations will be used throughout the paper.

$$(3.2) \quad \beta_e(k, \kappa, d, \varepsilon) = \frac{1}{\pi} \left(\ln \varepsilon + \ln 2 + \ln \frac{\pi}{d} \right) + \left(\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} \right),$$

$$(3.3) \quad \beta_i(k, \varepsilon) = \frac{\cot k}{k\varepsilon} + \frac{2 \ln 2}{\pi},$$

$$(3.4) \quad \beta(k, \kappa, d, \varepsilon) = \beta_e(k, \kappa, d, \varepsilon) + \beta_i(k, \varepsilon),$$

$$(3.5) \quad \tilde{\beta}(k, \varepsilon) = \frac{1}{(k \sin k)\varepsilon},$$

$$(3.6) \quad \rho(X, Y) = \frac{1}{\pi} \left[\ln |X - Y| + \ln \left(\left| \sin \left(\frac{\pi(X - Y)}{2} \right) \right| \right) \right. \\ \left. + \ln \left(\left| \sin \left(\frac{\pi(X + Y)}{2} \right) \right| \right) \right].$$

Remark 3.1. From the definition of $\zeta_n(k)$, it is clear that $\zeta_n(k) \neq 0$ for $k \in \mathbf{R}^+ \setminus B_{\kappa, \delta}$. In the above and throughout, the seemingly divergent expression

$$\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)}$$

should be understood in the following way where the series is convergent:

$$-\frac{i}{d} \frac{1}{\zeta_0(k)} + \sum_{n \neq 0} \left(\frac{1}{2\pi} \frac{1}{|n|} - \frac{i}{d} \frac{1}{\zeta_n(k)} \right).$$

Therefore, the scalar function $\beta_e(k, \kappa, d, \varepsilon)$ is well defined.

Remark 3.2. The real number $\delta = O(\varepsilon^{2\tau})$ denotes the distance from the Rayleigh cutoff frequencies. When $\tau = 0$ and $\delta = O(1)$, the frequency in $\mathbf{R}^+ \setminus B_{\kappa, \delta}$ is then far away from the Rayleigh cutoff frequencies. On the other hand, if $0 < \tau < 1$, a frequency in $\mathbf{R}^+ \setminus B_{\kappa, \delta}$ can be close to the Rayleigh cutoff frequencies. It should be pointed out that the assumption $0 \leq \tau < 1$ is essential for a uniform asymptotic expansion of the Green’s function, which is given in Lemma 3.1 below. The treatment for $\tau > 1$ would require more delicate asymptotic analysis. A promising approach is to use a new Green’s function that converges rapidly around the Rayleigh cutoff frequencies [12].

The asymptotic expansions for the kernels G_ε^e , G_ε^i , and \tilde{G}_ε^i are given in the following lemma.

LEMMA 3.1. *Let $\kappa \in (-\pi/d, \pi/d]$ and $k \in \mathbf{R}^+ \setminus B_{\kappa, \delta}$, where $\delta = O(\varepsilon^{2\tau})$ with $0 \leq \tau < 1$. If $k\varepsilon \ll 1$, then*

$$(3.7) \quad G_\varepsilon^e(X, Y) = G_\varepsilon^e(X, Y; \kappa) = \beta_e(k, \kappa, d, \varepsilon) + \frac{1}{\pi} \ln |X - Y| + r_1^\varepsilon(X, Y),$$

$$(3.8) \quad G_\varepsilon^i(X, Y) = \beta_i(k, \varepsilon) + \frac{1}{\pi} \left[\ln \left(\left| \sin \left(\frac{\pi(X+Y)}{2} \right) \right| \right) + \ln \left(\left| \sin \left(\frac{\pi(X-Y)}{2} \right) \right| \right) \right] + r_2^\varepsilon(X, Y),$$

$$(3.9) \quad \tilde{G}_\varepsilon^i(X, Y) = \tilde{\beta}(k, \varepsilon) + \tilde{\rho}_\infty(X, Y).$$

Here $r_1^\varepsilon(X, Y)$, $r_2^\varepsilon(X, Y)$, and $\tilde{\rho}_\infty(X, Y)$ are bounded functions with

$$r_1^\varepsilon \sim O(k\varepsilon^{1-\tau}), \quad r_2^\varepsilon \sim O((k\varepsilon)^2), \quad \text{and} \quad \tilde{\rho}_\infty \sim O(e^{-1/\varepsilon})$$

for all $X, Y \in (0, 1)$.

Proof. The asymptotic expansion for the kernels G_ε^i and \tilde{G}_ε^i has been derived in [23]. See also Appendix A. To obtain an asymptotic expansion for the kernel G_ε^e , let us consider the case when $\kappa = 0$. The case of $\kappa \neq 0$ can be calculated in a similar fashion. Using the Taylor expansion, we have

$$\begin{aligned} & \sum_{|\kappa_n| > k} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\ &= -\frac{id}{2\pi} \sum_{|\kappa_n| > k} \frac{1}{|n| \sqrt{1 - (kd/2\pi n)^2}} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} \\ &= -\frac{id}{2\pi} \sum_{|\kappa_n| > k} \frac{1}{|n|} \left(1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} \left(\frac{kd}{2\pi n} \right)^{2m} \right) e^{i\frac{2\pi n}{d} \varepsilon(X-Y)}. \end{aligned}$$

Then it follows that

$$\begin{aligned}
 & \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\
 &= \sum_{|\kappa_n| < k} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} + \sum_{|\kappa_n| > k} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\
 &= -\frac{id}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} + \frac{id}{2\pi} \sum_{|\kappa_n| < k, n \neq 0} \frac{1}{|n|} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} \\
 &\quad + \sum_{|\kappa_n| < k} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\
 (3.10) \quad & - \frac{id}{2\pi} \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} \sum_{|\kappa_n| > k} \left(\frac{kd}{2\pi n} \right)^{2m} \frac{1}{|n|} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)}.
 \end{aligned}$$

Notice that (cf. [19])

$$\begin{aligned}
 & -\sum_{n \neq 0} \frac{1}{|n|} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} = \ln \left(4 \sin^2 \frac{\pi \varepsilon(X-Y)}{d} \right), \\
 & \sum_{|\kappa_n| < k, n \neq 0} \frac{1}{|n|} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} = \sum_{|\kappa_n| < k, n \neq 0} \frac{1}{|n|} + O(k\varepsilon(X-Y)), \\
 & \sum_{|\kappa_n| < k} \frac{1}{\zeta_n(k)} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} = \sum_{|\kappa_n| < k} \frac{1}{\zeta_n(k)} + O(k\varepsilon^{1-\tau}(X-Y)) \quad \text{for } k \in \mathbf{R}^+ \setminus B_{\kappa, \delta}.
 \end{aligned}$$

On the other hand, for $m \geq 1$,

$$\begin{aligned}
 & \sum_{|\kappa_n| > k} \frac{1}{|n|^{2m+1}} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} \\
 &= \sum_{n \neq 0} \frac{1}{|n|^{2m+1}} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} - \sum_{|\kappa_n| < k, n \neq 0} \frac{1}{|n|^{2m+1}} e^{i\frac{2\pi n}{d} \varepsilon(X-Y)} \\
 &= \sum_{n \neq 0} \frac{1}{|n|^{2m+1}} + O(\varepsilon^{2m}(X-Y)^{2m} \ln(\varepsilon(X-Y))) \\
 &\quad - \sum_{|\kappa_n| < k, n \neq 0} \frac{1}{|n|^{2m+1}} + O(k\varepsilon(X-Y)) \\
 &= \sum_{|\kappa_n| > k} \frac{1}{|n|^{2m+1}} + O(k\varepsilon(X-Y)).
 \end{aligned}$$

Here the $O(k\varepsilon(X-Y))$ term is independent of m .

Substituting the above into (3.10), we have

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\ &= \frac{id}{2\pi} \ln \left(4 \sin^2 \frac{\pi \varepsilon(X-Y)}{d} \right) + \frac{id}{2\pi} \sum_{|\kappa_n| < k, n \neq 0} \frac{1}{|n|} + \sum_{|\kappa_n| < k} \frac{1}{\zeta_n(k)} \\ & \quad - \frac{id}{2\pi} \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} \sum_{|\kappa_n| > k} \left(\frac{kd}{2\pi} \right)^{2m} \frac{1}{|n|^{2m+1}} + O(k\varepsilon^{1-\tau}(X-Y)). \end{aligned}$$

Using the relation

$$\sum_{|\kappa_n| > k} \frac{1}{|n| \sqrt{1 - (kd/2\pi n)^2}} - \frac{1}{|n|} = \sum_{|\kappa_n| > k} \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} \left(\frac{kd}{2\pi} \right)^{2m} \frac{1}{|n|^{2m+1}},$$

we get

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\ &= \frac{id}{2\pi} \ln \left(4 \sin^2 \frac{\pi \varepsilon(X-Y)}{d} \right) + \left(\frac{id}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} + \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} \right) + O(k\varepsilon^{1-\tau}(X-Y)). \end{aligned}$$

Therefore, the desired asymptotic expansion follows by noting that

$$G_\varepsilon^e(X, Y) = -\frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)}. \quad \square$$

Let $\rho(X, Y)$ and $\tilde{\rho}_\infty(X, Y)$ be defined in (3.7) and (3.9), respectively. Set

$$\rho_\infty(X, Y) = r_1^\varepsilon(X, Y) + r_2^\varepsilon(X, Y),$$

where $r_1^\varepsilon(X, Y)$ and $r_2^\varepsilon(X, Y)$ are given in (3.7) and (3.8), respectively. We now introduce three integral operators corresponding to the Schwarz kernels $\rho(X, Y)$, $\rho_\infty(X, Y)$, and $\tilde{\rho}_\infty(X, Y)$:

$$\begin{aligned} (K\varphi)(X) &= \int_0^1 \rho(X, Y)\varphi(Y)dY, \quad X \in (0, 1), \\ (K_\infty\varphi)(X) &= \int_0^1 \rho_\infty(X, Y)\varphi(Y)dY, \quad X \in (0, 1), \\ (\tilde{K}_\infty\varphi)(X) &= \int_0^1 \tilde{\rho}_\infty(X, Y)\varphi(Y)dY, \quad X \in (0, 1). \end{aligned}$$

We also define the operator $P : V_1 \rightarrow V_2$ by

$$P\varphi(X) = \langle \varphi, 1 \rangle 1,$$

where 1 is a function defined on the interval (0, 1) and is equal to one, therein. We will use this notation in the following. One can easily check that $1 \in V_2$. Thus the above definition is valid.

LEMMA 3.2. Let $\kappa \in (-\pi/d, \pi/d]$ and $k \in \mathbf{R}^+ \setminus B_{\kappa, \delta}$, where $\delta = O(\varepsilon^{2\tau})$ with $0 \leq \tau < 1$.

(1) The operator $T^e + T^i$ admits the following decomposition:

$$T^e + T^i = \beta P + K + K_\infty.$$

Moreover, K_∞ is bounded from V_1 to V_2 with operator norm $\|K_\infty\| \lesssim \varepsilon^{1-\tau}$ uniformly for bounded k .

(2) The operator \tilde{T}^i admits the following decomposition:

$$\tilde{T}^i = \tilde{\beta} P + \tilde{K}_\infty.$$

Moreover, \tilde{K}_∞ is bounded from V_1 to V_2 with operator norm $\|\tilde{K}_\infty\| \lesssim e^{-1/\varepsilon}$ uniformly for bounded k .

(3) The operator K is bounded from V_1 to V_2 with a bounded inverse. Moreover,

$$\alpha := \langle K^{-1}1, 1 \rangle_{L^2(0,1)} \neq 0.$$

The proof of (1) and (2) follows directly from the definition of the operators T^e , T^i , and \tilde{T}^i in (2.6)–(2.8) and the asymptotic expansions of their kernels (see Lemma 3.1). The proof of (3) can be found in Theorem 4.1 and Lemma 4.2 of [14].

3.3. Asymptotic expansion of the solution to the system (2.9). Define

$$\mathbb{P} = \begin{bmatrix} \beta P & \tilde{\beta} P \\ \tilde{\beta} P & \beta P \end{bmatrix}, \quad \mathbb{K}_\infty = \begin{bmatrix} K_\infty & \tilde{K}_\infty \\ \tilde{K}_\infty & K_\infty \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f/\varepsilon \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbb{L} = K\mathbb{I} + \mathbb{K}_\infty.$$

Then from the decomposition of the operators in Lemma 3.2, we may rewrite the system of integral equations (2.9) as

$$(3.11) \quad (\mathbb{P} + \mathbb{L})\varphi = \mathbf{f}.$$

Next, we derive the asymptotic expansion of the solution φ . By Lemma 3.2, it is also easy to see that \mathbb{L} is invertible for sufficiently small ε . Applying the Neumann series yields

$$\mathbb{L}^{-1} = (K\mathbb{I} + \mathbb{K}_\infty)^{-1} = \left(\sum_{j=0}^{\infty} (-1)^j (K^{-1}\mathbb{K}_\infty)^j \right) K^{-1} = K^{-1}\mathbb{I} + O(k\varepsilon^{1-\tau}).$$

Therefore, the following lemma follows.

LEMMA 3.3. Let $\mathbf{e}_1 = [1, 0]^T$ and $\mathbf{e}_2 = [0, 1]^T$. Then

$$\mathbb{L}^{-1}\mathbf{e}_1 = K^{-1}1 \cdot \mathbf{e}_1 + O(k\varepsilon^{1-\tau}), \quad \mathbb{L}^{-1}\mathbf{e}_2 = K^{-1}1 \cdot \mathbf{e}_2 + O(k\varepsilon^{1-\tau})$$

and

$$(3.12) \quad \langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1 \rangle = \alpha + O(k\varepsilon^{1-\tau}), \quad \langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2 \rangle = O(k\varepsilon^{1-\tau}).$$

The following identities are proved in [23].

LEMMA 3.4. Let $\mathbf{e}_1 = [1, 0]^T$ and $\mathbf{e}_2 = [0, 1]^T$. Then

$$\langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2 \rangle, \quad \langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1 \rangle.$$

By applying \mathbb{L}^{-1} on both sides of (3.11), we see that

$$(3.13) \quad \mathbb{L}^{-1} \mathbb{P} \varphi + \varphi = \mathbb{L}^{-1} \mathbf{f}.$$

Note that since

$$\mathbb{P} \varphi = \beta \langle \varphi, \mathbf{e}_1 \rangle \mathbf{e}_1 + \beta \langle \varphi, \mathbf{e}_2 \rangle \mathbf{e}_2 + \tilde{\beta} \langle \varphi, \mathbf{e}_2 \rangle \mathbf{e}_1 + \tilde{\beta} \langle \varphi, \mathbf{e}_1 \rangle \mathbf{e}_2,$$

the above operator equation can be written as

$$(3.14) \quad \beta \langle \varphi, \mathbf{e}_1 \rangle \mathbb{L}^{-1} \mathbf{e}_1 + \beta \langle \varphi, \mathbf{e}_2 \rangle \mathbb{L}^{-1} \mathbf{e}_2 + \tilde{\beta} \langle \varphi, \mathbf{e}_2 \rangle \mathbb{L}^{-1} \mathbf{e}_1 + \tilde{\beta} \langle \varphi, \mathbf{e}_1 \rangle \mathbb{L}^{-1} \mathbf{e}_2 + \varphi = \mathbb{L}^{-1} \mathbf{f}.$$

By taking the inner product of (3.14) with \mathbf{e}_1 and \mathbf{e}_2 , respectively, it follows that

$$(3.15) \quad (\mathbb{M} + \mathbb{I}) \begin{bmatrix} \langle \varphi, \mathbf{e}_1 \rangle \\ \langle \varphi, \mathbf{e}_2 \rangle \end{bmatrix} = \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix},$$

where the matrix \mathbb{M} is defined as

$$(3.16) \quad \mathbb{M} := \beta \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle & \langle \mathbb{L}^{-1} \mathbf{e}_2, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_2 \rangle & \langle \mathbb{L}^{-1} \mathbf{e}_2, \mathbf{e}_2 \rangle \end{bmatrix} + \tilde{\beta} \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{e}_2, \mathbf{e}_1 \rangle & \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{e}_2, \mathbf{e}_2 \rangle & \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_2 \rangle \end{bmatrix}.$$

Therefore,

$$(3.17) \quad \begin{bmatrix} \langle \varphi, \mathbf{e}_1 \rangle \\ \langle \varphi, \mathbf{e}_2 \rangle \end{bmatrix} = (\mathbb{M} + \mathbb{I})^{-1} \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix}.$$

Substituting into (3.13) yields

$$(3.18) \quad \varphi = \mathbb{L}^{-1} \mathbf{f} - \begin{bmatrix} \mathbb{L}^{-1} \mathbf{e}_1 & \mathbb{L}^{-1} \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \beta & \tilde{\beta} \\ \tilde{\beta} & \beta \end{bmatrix} (\mathbb{M} + \mathbb{I})^{-1} \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix}.$$

From Lemma 3.4, it is observed that

$$\mathbb{M} = \left(\beta + \tilde{\beta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle & \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_2 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_2 \rangle & \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle \end{bmatrix}.$$

A straightforward calculation shows that the eigenvalues of $\mathbb{M} + \mathbb{I}$ are

$$(3.19) \quad \lambda_1(k; \kappa, d, \varepsilon) = 1 + (\beta + \tilde{\beta}) (\langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle + \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_2 \rangle),$$

$$(3.20) \quad \lambda_2(k; \kappa, d, \varepsilon) = 1 + (\beta - \tilde{\beta}) (\langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 \rangle - \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_2 \rangle),$$

and the associated eigenvectors are $[1 \quad 1]^T$ and $[1 \quad -1]^T$. For simplicity of notation, we define two scalar functions

$$(3.21) \quad p(k; \kappa, d, \varepsilon) := \varepsilon \lambda_1(k; \kappa, d, \varepsilon) \quad \text{and} \quad q(k; \kappa, d, \varepsilon) := \varepsilon \lambda_2(k; \kappa, d, \varepsilon).$$

We also define

$$(3.22) \quad \gamma(k, \kappa, d) = \frac{1}{\pi} \left(3 \ln 2 + \ln \frac{\pi}{d} \right) + \left(\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} \right).$$

Then a combination of (3.19)–(3.21), the expressions (3.2)–(3.5) for β and $\tilde{\beta}$, and Lemma 3.3 yields

$$(3.23) \quad p(k; \kappa, d, \varepsilon) = \varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k \sin k} + \varepsilon \gamma(k, \kappa, d) + \frac{1}{\pi} \varepsilon \ln \varepsilon \right] (\alpha + O(k\varepsilon^{1-\tau}))$$

and

$$(3.24) \quad q(k; \kappa, d, \varepsilon) = \varepsilon + \left[\frac{\cot k}{k} - \frac{1}{k \sin k} + \varepsilon \gamma(k, \kappa, d) + \frac{1}{\pi} \varepsilon \ln \varepsilon \right] (\alpha + O(k\varepsilon^{1-\tau})).$$

LEMMA 3.5. *Let $\kappa = k \sin \theta$, $\delta = O(\varepsilon^{2\tau})$, where $0 \leq \tau < 1$, and $k \in \mathbf{R}^+ \setminus B_{\kappa, \delta}$ be bounded and not an eigenvalue of the scattering operator. Then the following asymptotic expansion holds for the solution φ of (2.9) in $V_1 \times V_1$:*

$$\begin{aligned} \varphi &= K^{-1} \mathbf{1} \cdot \left[\kappa \cdot O(1) \cdot \mathbf{e}_1 + \frac{\alpha}{p} (\mathbf{e}_1 + \mathbf{e}_2) + \frac{\alpha}{q} (\mathbf{e}_1 - \mathbf{e}_2) \right] \\ &\quad + \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) \cdot O(k\varepsilon^{1-\tau}) + O(k\varepsilon^{1-\tau}). \end{aligned}$$

Moreover,

$$(3.25) \quad \begin{bmatrix} \langle \varphi, \mathbf{e}_1 \rangle \\ \langle \varphi, \mathbf{e}_2 \rangle \end{bmatrix} = \left[\alpha + O(\varepsilon^{1-\tau}) \right] \left(\frac{1}{p} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{q} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right).$$

Proof. The matrix $\mathbb{M} + \mathbb{I}$ has two eigenvalues λ_1 and λ_2 (see (3.19)–(3.20)), which are associated with the eigenvectors $[1 \ 1]^T$ and $[1 \ -1]^T$, respectively. Thus

$$(\mathbb{M} + \mathbb{I})^{-1} = \frac{1}{2\lambda_1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2\lambda_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

By substituting the above into (3.18) and (3.17), it follows that

$$\begin{bmatrix} \langle \varphi, \mathbf{e}_1 \rangle \\ \langle \varphi, \mathbf{e}_2 \rangle \end{bmatrix} = \frac{1}{2\lambda_1} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 + \mathbf{e}_2 \rangle \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2\lambda_2(k, \varepsilon)} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 - \mathbf{e}_2 \rangle \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{aligned} \varphi &= \mathbb{L}^{-1} \mathbf{f} - \frac{1}{2\lambda_1} \begin{bmatrix} \mathbb{L}^{-1} \mathbf{e}_1 & \mathbb{L}^{-1} \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \beta & \tilde{\beta} \\ \tilde{\beta} & \beta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix} \\ &\quad - \frac{1}{2\lambda_2(k, \varepsilon)} \begin{bmatrix} \mathbb{L}^{-1} \mathbf{e}_1 & \mathbb{L}^{-1} \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \beta & \tilde{\beta} \\ \tilde{\beta} & \beta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix}. \end{aligned}$$

A further calculation yields

$$\begin{aligned} \varphi &= \mathbb{L}^{-1} \mathbf{f} + \frac{1 - \lambda_1 / \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 \rangle}{2\lambda_1} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 + \mathbf{e}_2 \rangle \cdot (\mathbb{L}^{-1} \mathbf{e}_1 + \mathbb{L}^{-1} \mathbf{e}_2) \\ &\quad + \frac{1 - \lambda_2(k, \varepsilon) / \langle \mathbb{L}^{-1} \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 \rangle}{2\lambda_2(k, \varepsilon)} \langle \mathbb{L}^{-1} \mathbf{f}, \mathbf{e}_1 - \mathbf{e}_2 \rangle \cdot (\mathbb{L}^{-1} \mathbf{e}_1 - \mathbb{L}^{-1} \mathbf{e}_2). \end{aligned}$$

On the other hand, it is easy to check that

$$\mathbf{f} = \frac{1}{\varepsilon} 2 \cdot \mathbf{e}_1 + \kappa \cdot O(1) \cdot \mathbf{e}_1 \quad \text{in } V_2 \times V_2.$$

Thus we have from Lemma 3.3 that

$$\mathbb{L}^{-1} \mathbf{f} = \frac{1}{\varepsilon} [2 + \kappa \cdot O(\varepsilon)] [K^{-1} \mathbf{1} \cdot \mathbf{e}_1 + O(\varepsilon^{1-\tau})].$$

Combined with Lemma 3.4, it follows that

$$\begin{aligned} \varepsilon\varphi &= [2 + \kappa \cdot O(\varepsilon)] K^{-1}\mathbf{1} \cdot \mathbf{e}_1 + O(\varepsilon^{1-\tau}) \\ &\quad + \frac{1 - \lambda_1/(\alpha + O(\varepsilon^{1-\tau}))}{2\lambda_1} \cdot [2\alpha + O(\varepsilon^{1-\tau})] \cdot [K^{-1}\mathbf{1} \cdot (\mathbf{e}_1 + \mathbf{e}_2) + O(\varepsilon^{1-\tau})] \\ &\quad + \frac{1 - \lambda_2/(\alpha + O(\varepsilon^{1-\tau}))}{2\lambda_2} \cdot [2\alpha + O(\varepsilon^{1-\tau})] \cdot [K^{-1}\mathbf{1} \cdot (\mathbf{e}_1 - \mathbf{e}_2) + O(\varepsilon^{1-\tau})] \\ &= \kappa \cdot O(\varepsilon) \cdot K^{-1}\mathbf{1} \cdot \mathbf{e}_1 + \frac{\alpha}{\lambda_1} [K^{-1}\mathbf{1} \cdot (\mathbf{e}_1 + \mathbf{e}_2) + O(\varepsilon^{1-\tau})] \\ &\quad + \frac{\alpha}{\lambda_2} [K^{-1}\mathbf{1} \cdot (\mathbf{e}_1 - \mathbf{e}_2) + O(\varepsilon^{1-\tau})] + O(\varepsilon^{1-\tau}). \end{aligned}$$

Similarly, we can deduce (3.25). This completes the proof of the lemma. \square

3.4. Asymptotic expansion of the solution to the scattering problem.

Denote the far-field zones above and below the slab by $\Omega_1^+ := \{x \mid x_2 > 2\}$ and $\Omega_1^- := \{x \mid x_2 < -1\}$, respectively.

LEMMA 3.6. *The scattered field in Ω_1^\pm has the following asymptotic expansion:*

$$u_\varepsilon^s(x) = -\varepsilon\alpha g^e(x, (0, 1)) \cdot \left(\frac{1}{p} \pm \frac{1}{q}\right) + O(\varepsilon^{2-\tau}) \cdot \left(\frac{1}{p} \pm \frac{1}{q}\right),$$

where p and q are defined in (3.23) and (3.24), respectively.

Proof. In Ω_1^+ , the scattered field

$$u_\varepsilon^s(x) = \int_{\Gamma_\varepsilon^+} g^e(x, y) \frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y.$$

Recalling that

$$\frac{\partial u_\varepsilon}{\partial \nu}(x_1, 1) = -\varphi_1\left(\frac{x_1}{\varepsilon}\right),$$

we have

$$u_\varepsilon^s(x) = - \int_{\Gamma_\varepsilon^+} g^e(x, (y_1, 1)) \varphi_1\left(\frac{y_1}{\varepsilon}\right) dy_1 = -\varepsilon \int_0^1 g^e(x, (\varepsilon Y, 1)) \varphi_1(Y) dY.$$

By noting that

$$g^e(x, (\varepsilon Y, 1)) = g^e(x, (0, 1)) (1 + O(\varepsilon)) \quad \text{for } x \in \Omega_1^+,$$

and using the asymptotic expansion for $\langle \varphi, \mathbf{e}_1 \rangle$ in Lemma 3.5, we arrive at the desired formula. The scattered field in Ω_1^- can be obtained analogously. \square

Next we consider the wavefield in the slits. Since u_ε is quasi-periodic, we restrict the discussion to the reference slit $S_\varepsilon^{(0)}$ only. Observe that u_ε satisfies

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 & \text{in } S_\varepsilon^{(0)}, \\ \frac{\partial u_\varepsilon}{\partial x_1} = 0 & \text{on } x_1 = 0, x_1 = \varepsilon. \end{cases}$$

Since $k\varepsilon \ll 1$, in light of the boundary condition on the slit walls, we may expand u_ε as the sum of waveguide modes as follows:

$$(3.26) \quad u_\varepsilon(x) = a_0 \cos kx_2 + b_0 \cos k(1 - x_2) + \sum_{m \geq 1} \left(a_m e^{-k_2^{(m)} x_2} + b_m e^{-k_2^{(m)} (1-x_2)} \right) \cos \frac{m\pi x_1}{\varepsilon},$$

where $k_2^{(m)} = \sqrt{(m\pi/\varepsilon)^2 - k^2}$.

LEMMA 3.7. *The wavefield in the slit region*

$$S_\varepsilon^{(0),int} := \{x \in S_\varepsilon^{(0)} \mid x_2 \gg \varepsilon, 1 - x_2 \gg \varepsilon\}$$

has the following asymptotic expansion:

$$u_\varepsilon(x) = \left[\alpha + O(\varepsilon^{1-\tau}) \right] \left[\frac{\cos(kx_2)}{k \sin k} \left(\frac{1}{p} + \frac{1}{q} \right) + \frac{\cos(k(1-x_2))}{k \sin k} \left(\frac{1}{p} - \frac{1}{q} \right) \right] + O(e^{-1/\varepsilon}).$$

Proof. Taking the derivative of (3.26) and evaluating on the slit apertures, one has

$$(3.27) \quad \frac{\partial u_\varepsilon}{\partial x_2}(x_1, 1) = -a_0 k \sin k + \sum_{m \geq 1} \left(-a_m e^{-k_2^{(m)}} + b_m \right) k_2^{(m)} \cos \frac{m\pi x_1}{\varepsilon},$$

$$(3.28) \quad \frac{\partial u_\varepsilon}{\partial x_2}(x_1, 0) = b_0 k \sin k + \sum_{m \geq 1} \left(-a_m + b_m e^{-k_2^{(m)}} \right) k_2^{(m)} \cos \frac{m\pi x_1}{\varepsilon}.$$

Therefore,

$$\begin{aligned} -a_0 k \sin k &= \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon^+} \frac{\partial u_\varepsilon}{\partial x_2}(x_1, 1) dx_1 = - \int_0^1 \varphi_1(X) dX = - \left[\alpha + O(\varepsilon^{1-\tau}) \right] \left(\frac{1}{p} + \frac{1}{q} \right), \\ b_0 k \sin k &= \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon^-} \frac{\partial u_\varepsilon}{\partial x_2}(x_1, 0) dx_1 = \int_0^1 \varphi_2(X) dX = \left[\alpha + O(\varepsilon^{1-\tau}) \right] \left(\frac{1}{p} - \frac{1}{q} \right). \end{aligned}$$

Consequently

$$(3.29) \quad a_0 = \frac{1}{k \sin k} \left[\alpha + O(\varepsilon^{1-\tau}) \right] \left(\frac{1}{p} + \frac{1}{q} \right), \quad b_0 = \frac{1}{k \sin k} \left[\alpha + O(\varepsilon^{1-\tau}) \right] \left(\frac{1}{p} - \frac{1}{q} \right).$$

For $m \geq 1$, the coefficients a_m and b_m can be obtained similarly by taking the inner product of (3.27) and (3.28) with $\cos \frac{m\pi x_1}{\varepsilon}$. Then a direct estimate leads to

$$(3.30) \quad |a_m| \leq C/\sqrt{m}, \quad |b_m| \leq C/\sqrt{m} \quad \text{for } m \geq 1,$$

where C is some positive constant independent of ε , k , and m . The proof is complete by substituting (3.29) and (3.30) into (3.26). \square

We now consider the wave field on the apertures Γ_ε^\pm of the reference slit $S_\varepsilon^{(0)}$. Define

$$(3.31) \quad h(X) = \frac{1}{\pi} \int_0^1 \ln |X - Y| (K^{-1}1)(Y) dY.$$

Let us rewrite $\beta_\varepsilon(k, \kappa, d, \varepsilon) = \frac{1}{\pi} \ln \varepsilon + \bar{\beta}_\varepsilon(k, \kappa, d)$, where

$$(3.32) \quad \bar{\beta}_\varepsilon(k, \kappa, d) := \frac{1}{\pi} \left(\ln 2 + \ln \frac{\pi}{d} \right) + \left(\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} \right).$$

LEMMA 3.8. *The following expansions hold:*

$$\begin{aligned} u_\varepsilon(x_1, 1) &= -\frac{1}{\pi} \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) \cdot \varepsilon \ln \varepsilon - \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) (\bar{\beta}_\varepsilon + h(x_1/\varepsilon)) \cdot \varepsilon + 2 \\ &\quad - \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) \cdot O(\varepsilon^{2-\tau} \ln \varepsilon) - \kappa \cdot O(\varepsilon) + O(\varepsilon^{2-\tau}) \end{aligned}$$

and

$$u_\varepsilon(x_1, 0) = -\frac{1}{\pi} \left(\frac{\alpha}{p} - \frac{\alpha}{q} \right) \cdot \varepsilon \ln \varepsilon - \left(\frac{\alpha}{p} - \frac{\alpha}{q} \right) (\bar{\beta}_e + h(x_1/\varepsilon)) \cdot \varepsilon - \left(\frac{\alpha}{p} - \frac{\alpha}{q} \right) \cdot O(\varepsilon^{2-\tau} \ln \varepsilon) - \kappa \cdot O(\varepsilon) + O(\varepsilon^{2-\tau})$$

on the slit apertures Γ^+ and Γ^- , respectively.

Proof. Recall that on Γ_ε^+ ,

$$u_\varepsilon(x) = \int_{\Gamma_\varepsilon^+} g_\varepsilon^e(x, y) \frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + u^i + u^r.$$

Let $x_1 = \varepsilon X$, $y_1 = \varepsilon Y$. We have

$$u_\varepsilon(\varepsilon X, 1) = - \int_0^1 G_\varepsilon^e(X, Y) \varepsilon \varphi_1(Y) dY + f(X).$$

Using Lemma 3.5 and the asymptotic expansion of $G_\varepsilon^e(X, Y)$ in Lemma 3.1, we obtain

$$\begin{aligned} u_\varepsilon(\varepsilon X, 1) &= -\varepsilon \beta_e \left(\alpha + O(\varepsilon^{1-\tau}) \right) \left(\frac{1}{p} + \frac{1}{q} \right) \\ &\quad - \frac{\varepsilon}{\pi} \left(\kappa \cdot O(1) + \frac{\alpha}{p} + \frac{\alpha}{q} \right) \int_0^1 \ln |X - Y| (K^{-1}1)(Y) dY \\ &\quad - \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) O(\varepsilon^{2-\tau}) + O(\varepsilon^{2-\tau}) + f(X). \end{aligned}$$

The desired expansion follows by noting (3.31) and (3.32). The wavefield on the lower aperture can be obtained similarly. \square

3.5. An overview of field enhancement and diffraction anomalies. From Lemmas 3.6 to 3.8, we observe that $p(k; \kappa, d, \varepsilon)$ and $q(k; \kappa, d, \varepsilon)$ are two key scalar functions for analyzing the anomalous behaviors of the diffracted wavefield. For instance, the wavefield will exhibit extraordinary enhancement for vanishing p or q . In addition, $\zeta_n(k)$, and, consequently, the function $\gamma(k, \kappa, d)$ (see (3.22)–(3.24)) has a branch cut at certain frequencies, and this may give rise to an anomalous diffracted field too. The following is a summary of several cases that we will explore in the rest of the paper.

- (i) $p(k; \kappa, d, \varepsilon) = 0$ or $q(k; \kappa, d, \varepsilon) = 0$ attain complex roots k with negative imaginary part and real part $\text{Re } k > |\kappa|$. Such k 's are called resonances and the corresponding modes are called quasi-modes or leaky modes. If the incident frequency coincides with the resonant frequency, then field enhancement will occur.
- (ii) $p(k; \kappa, d, \varepsilon) = 0$ or $q(k; \kappa, d, \varepsilon) = 0$ attain real roots k with $k < |\kappa|$. Such k 's are called real eigenvalues of the scattering operator, and the corresponding eigenmodes are called Rayleigh–Bloch surface bound states that are confined near the periodic structure. The surface bound-state modes can couple with nearby sources through near-field interaction, but not with the plane incident wave that is considered here.
- (iii) $p(k; \kappa, d, \varepsilon) = 0$ or $q(k; \kappa, d, \varepsilon) = 0$ attain real roots k with $k > |\kappa|$. In such a scenario, the periodic slab structures possesses a certain finite bound state embedded in the continuum states (or a point spectrum embedded in the continuous spectrum).

- (iv) The function $\gamma = \gamma(k, \kappa, d)$ has a branch cut at the triplet (k, κ, d) such that $k = |\kappa + 2\pi n/d|$ and $\zeta_n(k) = 0$. This corresponds to the Rayleigh anomaly, where the propagating mode $e^{i\kappa_n x_1 \pm i\zeta_n x_2}$ becomes an evanescent mode or vice versa.

We investigate (i)–(ii) in section 4, and explore field enhancement in section 5 when the resonance frequency is close to the Rayleigh anomaly. The embedded eigenvalues (iii) are discussed briefly in section 6.

4. Resonances and eigenvalues away from Rayleigh cutoff frequencies.

4.1. The homogenous scattering problem. In order to obtain eigenvalues or resonances of the scattering problem, we consider the corresponding homogeneous problem when the incident wave $u^i = 0$. The solution k is either real valued or complex valued with a negative imaginary part. The former is called an eigenvalue and the latter is called a resonance. Here we focus on eigenvalues or resonances that are sufficiently away from the Rayleigh cutoff frequencies. So we assume that $\tau = 0$ in (3.1) such that $\delta := O(\varepsilon^{2\tau}) = O(1)$. In addition, it is natural to assume that $\delta < \pi/d$ so that $\mathbf{R} \setminus \mathbf{B}_{\kappa, \delta} \neq \emptyset$. On the other hand, we only consider eigenvalues/resonances not in the high frequency regime. Therefore, we restrict ourselves to the domain of interest

$$D_{\kappa, \delta, M} := \{z : |z| \leq M\} \setminus B_{\kappa, \delta},$$

where $M > 0$ is a fixed constant.

As shown later on, for each $\kappa \in (-\pi/d, \pi/d]$, the eigenvalues and resonances in $D_{\kappa, \delta, M}$ lie in the vicinity of

$$k_{m,0} := m\pi \quad \text{for } m = 1, 2, \dots \text{ and } m\pi < M.$$

Therefore, we extend the asymptotic expansions of the boundary integral operators to a neighborhood of $k_{m,0}$ on the complex plane. More precisely, let $\delta_0 = \min\{\delta/2, \pi\}$. If $k_{m,0} \in D_{\kappa, \delta, M}$, we choose the disc with radius δ_0 centered at $k_{m,0}$ on the complex k -plane, which is denoted as $B_{\delta_0}(k_{m,0})$. We analytically extend the functions $\beta_e(k)$, $\beta_i(k)$, and $\tilde{\beta}(k)$, which are defined in (3.2)–(3.5) for real k , to the neighborhood of $k_{m,0}$, $B_{\delta_0}(k_{m,0})$. One can show that the asymptotic expansions for the kernels G_ε^e , G_ε^i , and \tilde{G}_ε^i given in Lemma 3.1 hold in $B_{\delta_0}(k_{m,0})$.

By virtue of (3.11), the homogeneous problem is equivalent to the operator equation

$$(\mathbb{P} + \mathbb{L})\varphi = 0 \quad \text{in } B_{\delta_0}(k_{m,0}).$$

In light of (3.15), this reduces to

$$(\mathbb{M} + \mathbb{I}) \begin{bmatrix} \langle \varphi, \mathbf{e}_1 \rangle \\ \langle \varphi, \mathbf{e}_2 \rangle \end{bmatrix} = 0,$$

where the matrix \mathbb{M} is defined by (3.16). Note that the eigenvalues of $\mathbb{M} + \mathbb{I}$, λ_1 and λ_2 , are given by (3.19) and (3.20), thus the characteristic values of the operator-valued function $\mathbb{P} + \mathbb{L}$ are the roots of the two analytic functions $\lambda_1(k; \kappa, d, \varepsilon)$ and $\lambda_2(k; \kappa, d, \varepsilon)$ or, equivalently, $p(k; \kappa, d, \varepsilon)$ and $q(k; \kappa, d, \varepsilon)$ as defined in (3.21).

LEMMA 4.1. *The resonances of the scattering problem (1.1)–(1.4) in $B_{\delta_0}(k_{m,0})$ are the roots of one of the analytic functions $p(k; \kappa, d, \varepsilon) = 0$ and $q(k; \kappa, d, \varepsilon) = 0$ with $\text{Im} k < 0$, and the eigenvalues are those with $\text{Im} k = 0$.*

LEMMA 4.2. For each $\kappa \in (-\pi/d, \pi/d]$, the roots of $p(k; \kappa, d, \varepsilon) = 0$ and $q(k; \kappa, d, \varepsilon) = 0$ in the domain $D_{\kappa, \delta, M}$ attain the following asymptotic expansion:

$$(4.1) \quad k_m = k_m(\kappa, d, \varepsilon) = m\pi + 2m\pi \left[\frac{1}{\pi} \varepsilon \ln \varepsilon + \left(\frac{1}{\alpha} + \gamma(m\pi, \kappa, d) \right) \varepsilon \right] + O(\varepsilon^2 \ln^2 \varepsilon)$$

for each integer m . Here γ is defined in (3.22).

Proof. Let $c(k) = \frac{\cot k}{k} + \frac{1}{k \sin k}$. From (3.23), it follows that

$$(4.2) \quad p(k; \kappa, d, \varepsilon) = \varepsilon + \left[c(k) + \varepsilon \gamma(k, \kappa, d) + \frac{1}{\pi} \varepsilon \ln \varepsilon \right] (\alpha + r(k, \varepsilon)) = 0,$$

where $\gamma(k, \kappa, d)$ is defined in (3.22), $r(k, \varepsilon)$ is analytic in $B_{\delta_0}(k_{m,0})$, and $r(k, \varepsilon) \sim O(\varepsilon)$. It is clear that the analytic function $c(k)$ attains a simple root $k_{m,0} = m\pi$ in $B_{\delta_0}(k_{m,0})$ for odd integers m . From Rouché’s theorem, we deduce that there is a simple root of $p(k, \varepsilon)$, which is denoted as k_m , close to $k_{m,0}$ if ε is sufficiently small.

To obtain the leading-order asymptotic terms of k_m , first let us consider the root for

$$(4.3) \quad p_1(k; \kappa, d, \varepsilon) := \varepsilon + \left[c(k) + \varepsilon \gamma(k, \kappa, d) + \frac{1}{\pi} \varepsilon \ln \varepsilon \right] \alpha = 0.$$

The Taylor expansion for $p_1(k, \varepsilon)$ at $k = k_{m,0}$ yields

$$(4.4) \quad p_1(k; \kappa, d, \varepsilon) = \varepsilon + \left[c'(k_{m,0})(k - k_{m,0}) + O(k - k_{m,0})^2 + \varepsilon \gamma(k_{m,0}) + \varepsilon \cdot O(k - k_{m,0}) + \frac{1}{\pi} \varepsilon \ln \varepsilon \right] \alpha.$$

A direct calculation gives $c'(k_{m,0}) = -\frac{1}{2m\pi}$. We can deduce that p_1 has a simple root $k_{m,1}$ close to $k_{m,0}$, and is given by

$$k_{m,1} = k_{m,0} + 2m\pi \left[\frac{1}{\pi} \varepsilon \ln \varepsilon + \left(\frac{1}{\alpha} + \gamma_0(k_{m,0}, \kappa, d) \right) \varepsilon \right] + O(\varepsilon^2 \ln^2 \varepsilon).$$

Next we show that $k_m - k_{m,1} = O(\varepsilon^2 \ln^2 \varepsilon)$ and the desired asymptotic expansion (4.5) follows. Note that

$$p(k; \kappa, d, \varepsilon) - p_1(k; \kappa, d, \varepsilon) = O(c(k) + \varepsilon \ln \varepsilon) r(k, \varepsilon)$$

and

$$p_1(k; \kappa, d, \varepsilon) = c(k)\alpha + O(\varepsilon \ln \varepsilon).$$

Hence, one can find a constant $C_m > 0$ such that

$$|p(k; \kappa, d, \varepsilon) - p_1(k; \kappa, d, \varepsilon)| < |p_1(k; \kappa, d, \varepsilon)|$$

for all k such that $|k - k_{m,1}| = C_m \varepsilon^2 \ln^2 \varepsilon$. By Rouché’s theorem, p has a simple root in the disc $\{k \mid |k - k_{m,1}| = C_m \varepsilon^2 \ln^2 \varepsilon\}$, which proves our claim.

Similarly, we obtain the root of $q(k; \kappa, d, \varepsilon) = 0$ in $B_{\delta_0}(k_{m,0})$ for even integers m . The arguments are the same as above and we omit them here. \square

4.2. Asymptotic expansions of resonances and eigenvalues. The formula (4.1) gives the asymptotic expansion of resonances and eigenvalues for the scattering problem (1.1)–(1.4). We now distinguish between resonances and eigenvalues. First, observe that

$$\operatorname{Im} \gamma(m\pi, \kappa, d) \neq 0 \quad \text{if } m\pi > |\kappa|.$$

Therefore, k_m attains a nonzero imaginary part, and k_m is a complex-valued resonance. We immediately have the following proposition.

PROPOSITION 4.3. *For each $\kappa \in (-\pi/d, \pi/d]$, if $m\pi > |\kappa|$ such that $\operatorname{Im} \gamma(m\pi, \kappa, d) \neq 0$, then there exists a resonance in the domain $D_{\kappa, \delta, M}$ for the scattering problem (1.1)–(1.4) with the following asymptotic expansion*

$$(4.5) \quad k_m = m\pi + 2m\pi \left[\frac{1}{\pi} \varepsilon \ln \varepsilon + \left(\frac{1}{\alpha} + \gamma(m\pi, \kappa, d) \right) \varepsilon \right] + O(\varepsilon^2 \ln^2 \varepsilon).$$

Here $\alpha = \langle K^{-1}1, 1 \rangle$.

Remark 4.1. In Proposition 4.3, the imaginary part of the resonance k_m is given by $2m\pi \cdot \operatorname{Im} \gamma(m\pi, \kappa, d) \cdot \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon)$, which is negative and is of order $O(\varepsilon)$. As the size of the period $d \rightarrow +\infty$, we see that

$$\operatorname{Im} \gamma(m\pi, \kappa, d) = -\frac{1}{d} \sum_{|\kappa_n| < m\pi} \frac{1}{\zeta_n(m\pi)} \rightarrow -\frac{1}{2\pi} \int_{-m\pi}^{m\pi} \frac{1}{\sqrt{(m\pi)^2 - t^2}} dt = -\frac{1}{2}.$$

This constant is consistent with the one obtained for the case of a single slit perforated in a slab of infinite length (cf. [23]).

Now if $m\pi < |\kappa|$ such that

$$\operatorname{Im} \gamma(m\pi, \kappa, d) = 0,$$

the imaginary part of the $O(\varepsilon)$ -term of k_m in (4.1) is zero. However, one cannot tell directly from the asymptotic expansion (4.1) whether the higher-order terms of k_m are real or complex valued. Instead, we resort to the variational formulation to verify that those k_m are indeed real eigenvalues.

Let us first recall some basic facts from [13]. Denote $\Omega_\varepsilon^{(0)} = \Omega^{(0)} \cap \Omega_\varepsilon$ and define the function space

$$H_{\kappa, d}^1(\Omega_\varepsilon^{(0)}) = \left\{ u : u \in H^1(\Omega_\varepsilon^{(0)}), u(d, x_2) = e^{i\kappa d} u(0, x_2), \frac{\partial u}{\partial x_1}(d, x_2) = e^{i\kappa d} \frac{\partial u}{\partial x_1}(0, x_2) \right\}.$$

We define a sesquilinear form

$$a(u, v) := \int_{\Omega_\varepsilon^{(0)}} \nabla u \cdot \nabla \bar{v} dx$$

on $H_{\kappa, d}^1(\Omega_\varepsilon^{(0)}) \times H_{\kappa, d}^1(\Omega_\varepsilon^{(0)})$ and denote

$$(u, v) = \int_{\Omega_\varepsilon^{(0)}} u \bar{v} dx.$$

Let $A(\kappa) = A(\kappa, d, \varepsilon)$ be the operator associated with the sesquilinear form a such that

$$(A(\kappa)u, v) = a(u, v)$$

for all $u, v \in H_{\kappa, d}^1(\Omega_\varepsilon^{(0)})$. The following statement holds (cf. [13]).

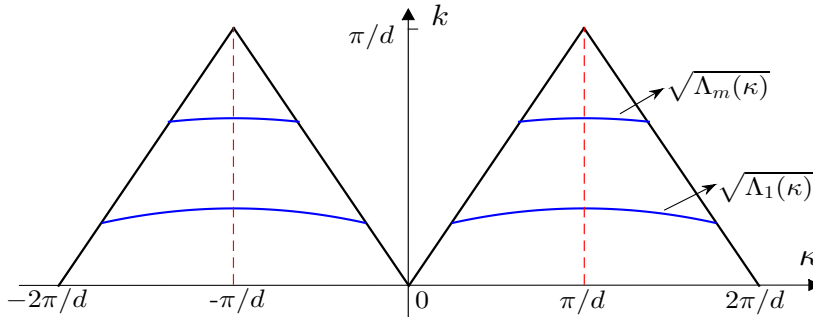


FIG. 3. Eigenvalues of the scattering operator $A(\kappa)$.

LEMMA 4.4.

1. (k, κ, u) is a solution to the homogeneous problem with $u^i = 0$ if and only if k^2 is an eigenvalue of $A(\kappa)$ and u is the associated eigenfunction.
2. $A(\kappa)$ is a positive self-adjoint operator.

For each positive integer m , let

$$\Lambda_m(\kappa) = \Lambda_m(\kappa, d, \varepsilon) = \inf_{V_m \in \mathcal{V}_m} \sup_{u \in V_m, u \neq 0} \frac{a(u, u)}{(u, u)},$$

where \mathcal{V}_m denotes the set of all m -dimensional subspaces of $H^1_{\kappa, d}(\Omega_\varepsilon)$. It is clear that for each fixed κ , $\Lambda_m(\kappa)$ is an increasing sequence as shown in Figure 3. We denote by $\mathcal{N}(\kappa) = \mathcal{N}(\kappa, d, \varepsilon)$ the number of eigenvalues Λ of $A(\kappa)$, counting their multiplicity, which are strictly less than $|\kappa|^2$.

LEMMA 4.5. *The following statements hold (cf. [13]):*

1. $\Lambda_m(\kappa) = \Lambda_m(-\kappa)$ for $\kappa \in [0, \pi/d]$.
2. For each fixed m , $\Lambda_m(\kappa)$ is a continuous function of κ .
3. For $\kappa \in (-\pi/d, \pi/d]$, if $k_{m,0} < |\kappa|$, then $\Lambda_m(\kappa) < \kappa^2$ and $\mathcal{N}(\kappa) \geq m$. In addition, $\Lambda_1, \Lambda_2, \dots, \Lambda_m$ are the first m eigenvalues of $A(\kappa)$.

Now let us first consider the special case when $\kappa = \pi/d$ and $m\pi < |\kappa|$. Then for real k in the neighborhood of $k_{m,0}$, the Green's function $G^\varepsilon_\varepsilon(X, Y; \pi/d)$ is a real-valued function, by noting that the n and $-(n + 1)$ terms in the following series expansion

$$\begin{aligned} G^\varepsilon_\varepsilon(X, Y; \pi/d) &= -\frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n \varepsilon(X-Y)} \\ &= -\frac{1}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{((2n+1)\pi/d)^2 - k^2}} e^{i(2n+1)\pi/d \varepsilon(X-Y)} \end{aligned}$$

are conjugate pairs. Therefore, all the terms in (3.19) and (3.20) are real. Consequently, $p(k; \pi/d, d, \varepsilon) = 0$ or $q(k; \pi/d, d, \varepsilon) = 0$ attains a real root k_m near $m\pi$, which is an eigenvalue of the scattering problem. The asymptotic expansion of the eigenvalue is given by (4.1). In particular, by Lemmas 4.4 and 4.5, we have

$$\Lambda_m(\pi/d, d, \varepsilon) = k_m^2(\pi/d, d, \varepsilon).$$

For $\kappa \neq \pi/d$, the continuity of $\Lambda_m(\kappa)$ and $k_m(\kappa)$ implies that $\Lambda_m(\kappa, d, \varepsilon) = k_m^2(\kappa, d, \varepsilon)$ as long as $m\pi < |\kappa|$, and $k_m(\kappa)$ is a real eigenvalue. In summary, we can draw the following conclusion.

PROPOSITION 4.6. For each $\kappa \in (-\pi/d, \pi/d]$, if $m\pi < |\kappa|$ such that $\text{Im}\gamma(m\pi, \kappa, d) = 0$, then there exist exactly m eigenvalues in the domain $D_{\kappa, \delta, M}$ for the scattering problem (1.1)–(1.4). Moreover, the following asymptotic expansion holds for each eigenvalue k_m :

$$(4.6) \quad k_m = m\pi + 2m\pi \left[\frac{1}{\pi} \varepsilon \ln \varepsilon + \left(\frac{1}{\alpha} + \gamma(m\pi, \kappa, d) \right) \varepsilon \right] + O(\varepsilon^2 \ln^2 \varepsilon).$$

Here $\alpha = \langle K^{-1}1, 1 \rangle$.

Noting that $k_m < |\kappa|$ and $\zeta_n(k_m)$ is a pure imaginary number for all n , we see that the associated eigenfunctions are Rayleigh–Bloch surface bound states that decay exponentially away from the slab structure.

Remark 4.2. Note that if the size of the period d satisfies the inequality

$$k_1(\pi/d, d, \varepsilon) \geq \pi/d,$$

then $\mathcal{N}(\pi/d) = 0$. Therefore, there is no eigenvalue for the scattering problem and only resonances exist. On the other hand, if

$$k_1(\pi/d, d, \varepsilon) < \pi/d,$$

then $\mathcal{N}(\pi/d, d, \varepsilon) \geq 1$. That is, in addition to resonances, there exists at least one eigenvalue which is near π .

4.3. Field enhancement at resonant frequencies. To investigate the field enhancement, recall that $p(k; \kappa, d, \varepsilon) := \varepsilon \lambda_1(k; \kappa, d, \varepsilon)$ and $q(k; \kappa, d, \varepsilon) := \varepsilon \lambda_2(k; \kappa, d, \varepsilon)$.

LEMMA 4.7. If $m\varepsilon \ll 1$, then at the resonant frequencies $k = \text{Re } k_m$,

$$p(k; \kappa, d, \varepsilon) = i \cdot \text{Im} \gamma(m\pi, \kappa, d) \cdot \alpha \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon)$$

and

$$q(k; \kappa, d, \varepsilon) = i \cdot \text{Im} \gamma(m\pi, \kappa, d) \cdot \alpha \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon),$$

where m is odd and even, respectively.

Proof. We expand $p(k, \varepsilon)$ and $q(k, \varepsilon)$ in the disk $\{|k - \text{Re } k_m| \leq \varepsilon |\ln \varepsilon|\}$. If m is odd, from the definition of p_1 in (4.3) and its expansion (4.4), it follows that

$$\begin{aligned} p(k; \kappa, d, \varepsilon) &= p_1(k; \kappa, d, \varepsilon) + O(\varepsilon^2 \ln \varepsilon) \\ &= p'_1(k_m)(k - k_m) + O(k - k_m)^2 + O(\varepsilon^2 \ln \varepsilon) \\ &= [\alpha c'(k_{m,0}) + O(\varepsilon \ln \varepsilon)] \cdot (k - k_m) + O(\varepsilon^2 \ln^2 \varepsilon) \\ &= \alpha c'(k_{m,0}) \cdot (k - k_m) + O(\varepsilon^2 \ln^2 \varepsilon) \\ &= -\frac{\alpha}{2m\pi} (k - \text{Re } k_m - i \text{Im } k_m) + O(\varepsilon^2 \ln^2 \varepsilon). \end{aligned}$$

Note that

$$\text{Im } k_m = 2m\pi \cdot \text{Im} \gamma(m\pi, \kappa, d) \cdot \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon).$$

We deduce that at the odd resonant frequencies $k = \text{Re } k_m$,

$$p(k; \kappa, d, \varepsilon) = i \cdot \text{Im} \gamma(m\pi, \kappa, d) \cdot \alpha \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon).$$

The calculations for $q(k; \kappa, d, \varepsilon)$ at the even resonant frequencies are the same. \square

The following proposition follows directly from Lemmas 3.5 and 4.7.

PROPOSITION 4.8. *At resonant frequencies, $\varphi \sim O(1/\varepsilon)$ in $V_1 \times V_1$ and $\langle \varphi, \mathbf{e}_i \rangle \sim O(1/\varepsilon), i = 1, 2$.*

We now investigate the field enhancement in far-field and near-field zones. From Lemma 3.6, the scattered field

$$u_\varepsilon^s(x) = -\varepsilon\alpha g^e(x, (0, 1)) \cdot \left(\frac{1}{p} \pm \frac{1}{q}\right) + O(\varepsilon^{2-\tau}) \cdot \left(\frac{1}{p} \pm \frac{1}{q}\right) \quad \text{for } x \in \Omega_1^\pm.$$

At the resonant frequencies $k = \text{Re } k_m$ when m is odd, an application of Lemma 4.7 yields

$$\frac{1}{p} = -\frac{i}{\text{Im } \gamma(m\pi, \kappa, d)\alpha\varepsilon} (1 + O(\varepsilon \ln^2 \varepsilon)).$$

Hence the scattered field

$$(4.7) \quad u_\varepsilon^s(x) = \frac{i}{\text{Im } \gamma(m\pi, \kappa, d)} \cdot g^e(x, (0, 1)) + O(\varepsilon \ln^2 \varepsilon) \quad \text{for } x \in \Omega_1^+.$$

It is seen that the scattering enhancement is of order $O(\varepsilon^{-1})$ compared to the $O(\varepsilon)$ order for the scattered field at nonresonant frequencies. In addition, the scattered field behaves as the radiating field of a periodic array of monopoles located at $\bigcup_{n=-\infty}^\infty (nd, 1)$. The same occurs at resonant frequencies $k = \text{Re } k_m$ when m is even by an application of Lemma 4.7.

Following similar calculations to the above, in the far-field zone Ω_1^- below the slits, the transmitted field is equivalent to the radiating field of an array of monopoles located at $\bigcup_{n=-\infty}^\infty (nd, 0)$, and is given by

$$u_\varepsilon^s(x) = -\varepsilon\alpha g^e(x, (0, 0)) \cdot \left(\frac{1}{p} - \frac{1}{q}\right) + O(\varepsilon^2) \cdot \left(\frac{1}{p} - \frac{1}{q}\right).$$

It follows that

$$(4.8) \quad u_\varepsilon^s(x) = \frac{i}{\text{Im } \gamma(m\pi, \kappa, d)} \cdot g^e(x, (0, 0)) + O(\varepsilon \ln^2 \varepsilon)$$

and

$$(4.9) \quad u_\varepsilon^s(x) = -\frac{i}{\text{Im } \gamma(m\pi, \kappa, d)} \cdot g^e(x, (0, 0)) + O(\varepsilon \ln^2 \varepsilon)$$

in Ω_1^- at the odd and even resonant frequencies, respectively. The transmission enhancement is of order $O(\varepsilon^{-1})$ at the resonant frequencies.

Remark 4.3. The amplitude of the scattered and transmitted fields also depends on the incident angle θ and the size of the period d . This is explicitly given by the scalar function $\text{Im } \gamma(m\pi, \kappa, d)$. The same holds true in the near field described below.

The shape of resonant wave modes in the slits and their enhancement orders are characterized in the following theorem.

THEOREM 4.9. *The wave field in the slit region*

$$S_\varepsilon^{(0),int} := \{x \in S_\varepsilon^{(0)} \mid x_2 \gg \varepsilon, 1 - x_2 \gg \varepsilon\}$$

has the following asymptotic expansion:

$$(4.10) \quad u_\varepsilon(x) = -\left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + O(1)\right) \cdot \frac{i \cdot \cos(k(x_2 - 1/2))}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \sin(k/2)} + \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)} + O(\varepsilon \ln \varepsilon),$$

$$(4.11) \quad u_\varepsilon(x) = \left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + O(1)\right) \cdot \frac{i \cdot \sin(k(x_2 - 1/2))}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \cos(k/2)} + \frac{\cos(k(x_2 - 1/2))}{\cos(k/2)} + O(\varepsilon \ln \varepsilon)$$

at the resonant frequencies $k = \operatorname{Re} k_m$, where m is odd and even, respectively.

Proof. By Lemma 3.7, in the region $S_\varepsilon^{(0), \text{int}}$,

$$\begin{aligned} u_\varepsilon(x) &= \left[\alpha + O(\varepsilon)\right] \left[\frac{\cos(kx_2)}{k \sin k} \left(\frac{1}{p} + \frac{1}{q}\right) + \frac{\cos(k(1-x_2))}{k \sin k} \left(\frac{1}{p} - \frac{1}{q}\right) \right] + O\left(e^{-1/\varepsilon}\right) \\ &= 2 \left[\alpha + O(\varepsilon)\right] \left[\frac{1}{p} \frac{\cos(k/2) \cos(k(x_2 - 1/2))}{k \sin k} - \frac{1}{q} \frac{\sin(k/2) \sin(k(x_2 - 1/2))}{k \sin k} \right] \\ &\quad + O\left(e^{-1/\varepsilon}\right). \end{aligned}$$

At resonant frequencies $k = \operatorname{Re} k_m$ when m is odd,

$$\frac{1}{p} = -\frac{i}{\operatorname{Im} \gamma(m\pi, \kappa, d)\alpha\varepsilon} (1 + O(\varepsilon \ln^2 \varepsilon)) \quad \text{and} \quad \frac{1}{q} = \frac{k \sin k}{(\cos k - 1)\alpha} (1 + O(\varepsilon \ln \varepsilon)).$$

Therefore,

$$\begin{aligned} u_\varepsilon(x) &= (1 + O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)) \left[-\frac{1}{\varepsilon} \cdot \frac{i \cos(k(x_2 - 1/2))}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \sin(k/2)} (1 + O(\varepsilon \ln^2 \varepsilon)) \right. \\ &\quad \left. + \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)} (1 + O(\varepsilon \ln \varepsilon)) \right] + O\left(e^{-1/\varepsilon}\right) \\ &= -\left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + O(1)\right) \cdot \frac{i}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \sin(k/2)} \cdot \cos(k(x_2 - 1/2)) \\ &\quad + \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)} + O(\varepsilon \ln \varepsilon). \end{aligned}$$

Similarly, at resonant frequencies $k = \operatorname{Re} k_m$ when m is even,

$$\frac{1}{p} = \frac{k \sin k}{(\cos k + 1)\alpha} (1 + O(\varepsilon \ln \varepsilon)) \quad \text{and} \quad \frac{1}{q} = -\frac{i}{\operatorname{Im} \gamma(m\pi, \kappa, d)\alpha\varepsilon} (1 + O(\varepsilon \ln^2 \varepsilon)),$$

and we obtain

$$\begin{aligned} u_\varepsilon(x) &= (1 + O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)) \left[\frac{\cos(k(x_2 - 1/2))}{k \cos(k/2)} (1 + O(\varepsilon \ln \varepsilon)) \right. \\ &\quad \left. + \frac{1}{\varepsilon} \cdot \frac{i \sin(k(x_2 - 1/2))}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \cos(k/2)} (1 + O(\varepsilon \ln^2 \varepsilon)) \right] + O\left(e^{-1/\varepsilon}\right) \\ &= \left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + O(1)\right) \cdot \frac{i}{\operatorname{Im} \gamma(m\pi, \kappa, d) \cdot k \cos(k/2)} \cdot \sin(k(x_2 - 1/2)) \\ &\quad + \frac{\cos(k(x_2 - 1/2))}{\cos(k/2)} + O(\varepsilon \ln \varepsilon). \quad \square \end{aligned}$$

Therefore, the enhancement due to resonance is of order $O(\varepsilon^{-1})$ in the slit. Moreover, the dominant modes in the slit take the simple form of $\cos(k(x_2 - 1/2))$ and $\sin(k(x_2 - 1/2))$ at odd and even resonant frequencies, respectively.

An application of Lemmas 3.8 and 4.7 leads to the enhanced field on the apertures of slits as stated below.

THEOREM 4.10. *Let function h and $\bar{\beta}_e$ be defined by (3.31) and (3.32), respectively. The wavefields on the apertures Γ_ε^+ and Γ_ε^- are*

$$(4.12) \quad u_\varepsilon(x_1, 1) = \frac{i}{\pi \cdot \text{Im} \gamma(m\pi, \kappa, d)} \cdot [\ln \varepsilon + (\bar{\beta}_e + h(x_1/\varepsilon))] + 2 + O(\varepsilon \ln^3 \varepsilon),$$

$$(4.13) \quad u_\varepsilon(x_1, 0) = \frac{i}{\pi \cdot \text{Im} \gamma(m\pi, \kappa, d)} \cdot [\ln \varepsilon + (\bar{\beta}_e + h(x_1/\varepsilon))] + O(\varepsilon \ln^3 \varepsilon)$$

at the resonant frequencies $k = \text{Re} k_m$ for odd m . At the resonant frequencies $k = \text{Re} k_m$ when m is even, the wavefields on the slit apertures Γ_ε^+ and Γ_ε^- are

$$(4.14) \quad u_\varepsilon(x_1, 1) = \frac{i}{\pi \cdot \text{Im} \gamma(m\pi, \kappa, d)} \cdot [\ln \varepsilon + (\bar{\beta}_e + h(x_1/\varepsilon))] + 2 + O(\varepsilon \ln^3 \varepsilon),$$

$$(4.15) \quad u_\varepsilon(x_1, 0) = -\frac{i}{\pi \cdot \text{Im} \gamma(m\pi, \kappa, d)} \cdot [\ln \varepsilon + (\bar{\beta}_e + h(x_1/\varepsilon))] + O(\varepsilon \ln^3 \varepsilon).$$

It is seen that the leading order of the resonant mode is a constant of order $O(\ln \varepsilon)$ along the slit apertures, and the enhancement due to the resonant scattering is of order $O(\varepsilon^{-1})$.

5. Field enhancement at resonant frequencies near the Rayleigh anomaly. The Rayleigh anomaly occurs at cutoff frequencies when $k = \pm(\kappa + 2\pi n/d)$ for some integer n such that $\zeta_n(k) = 0$. According to the Rayleigh–Bloch expansion (1.4), this corresponds to a grazing angle near which the propagating mode $e^{i\kappa_n x_1 \pm i\zeta_n x_2}$ becomes an evanescent mode and vice versa. In this section, we investigate the field enhancement at those resonant frequencies that are close to the Rayleigh cutoff frequencies, by assuming that $0 < \tau < 1$ in (3.1) and $\delta = O(\varepsilon^{2\tau}) \ll 1$.

We consider a pair (k^0, κ^0) which satisfies $k^0 = \kappa^0 + 2\pi n_0/d$ for some integer n_0 and k^0 is a cutoff frequency. For clarity of presentation, let us assume that $\kappa^0 \neq 0$ and $\kappa^0 \neq \pi/d$ so that only one mode among all diffracted orders turns into an evanescent mode near (k^0, κ^0) . However, the following derivations can be extended for the case when $\kappa^0 = 0, \pi/d$. Suppose that k is perturbed away from the cutoff frequency k^0 such that $kd = k^0d + \delta = \kappa^0d + 2\pi n_0 + \delta$, where $\delta = O(\varepsilon^{2\tau})$ and $0 < \tau < 1$. It is clear that $e^{i\kappa_{n_0} x_1 \pm i\zeta_n x_2}$ is a propagating mode if $\delta > 0$ and an evanescent mode if $\delta < 0$. A direct expansion yields

$$\frac{1}{d \cdot \zeta_{n_0}(k)} = \frac{1}{\sqrt{|\delta|}} e^{-\frac{1}{2}i \arg \delta} \left(\frac{1}{\sqrt{2k_0d}} + O(\delta) \right).$$

From the definition of γ in (3.22), we see that

$$\begin{aligned} \gamma(k, \kappa^0, d) &= \frac{1}{\pi} \left(3 \ln 2 + \ln \frac{\pi}{d} \right) + \left(\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n \neq n_0} \frac{1}{\zeta_n(k)} \right) \\ &\quad - \frac{i}{\sqrt{|\delta|}} e^{-\frac{1}{2}i \arg \delta} \left(\frac{1}{\sqrt{2k_0d}} + O(\delta) \right) \\ &=: \gamma_0 - \frac{i}{\sqrt{|\delta|}} e^{-\frac{1}{2}i \arg \delta} \left(\frac{1}{\sqrt{2k_0d}} + O(\delta) \right). \end{aligned}$$

Consequently, from (3.23) it follows that

$$\begin{aligned} p(k, \kappa^0, d\varepsilon) &= \varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k \sin k} + \varepsilon\gamma(k, \kappa^0, d) + \frac{1}{\pi}\varepsilon \ln \varepsilon \right] (\alpha + r(k, \varepsilon)), \\ &= \varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k \sin k} + \varepsilon\gamma_0 - \frac{i\varepsilon}{\sqrt{|\delta|}} e^{-\frac{1}{2}i \arg \delta} \left(\frac{1}{\sqrt{2k_0d}} + O(\delta) \right) + \frac{1}{\pi}\varepsilon \ln \varepsilon \right] (\alpha + r(k, \varepsilon)). \end{aligned}$$

Similarly, using (3.24) for $q(k, \kappa^0, d\varepsilon)$ yields

$$q(k, \kappa^0, d\varepsilon) = \varepsilon + \left[\frac{\cot k}{k} - \frac{1}{k \sin k} + \varepsilon\gamma_0 - \frac{i\varepsilon}{\sqrt{|\delta|}} e^{-\frac{1}{2}i \arg \delta} \left(\frac{1}{\sqrt{2k_0d}} + O(\delta) \right) + \frac{1}{\pi}\varepsilon \ln \varepsilon \right] (\alpha + s(k, \varepsilon)).$$

LEMMA 5.1. Assume that $kd = k^0d + \delta$, where $\delta = O(\varepsilon^{2\tau})$ for $0 < \tau < 1$. If $k = m\pi$, then for $\delta > 0$,

$$p(k; \kappa^0, d, \varepsilon) = -\frac{i\alpha}{\sqrt{2|C_0|k_0d}} \cdot \varepsilon^{1-\tau} + \min\{O(\varepsilon^{2-2\tau}), O(\varepsilon)\}$$

and

$$q(k; \kappa^0, d, \varepsilon) = -\frac{i\alpha}{\sqrt{2|C_0|k_0d}} \cdot \varepsilon^{1-\tau} + \min\{O(\varepsilon^{2-2\tau}), O(\varepsilon)\}$$

when m is odd and even, respectively. For $\delta < 0$,

$$p(k; \kappa^0, d, \varepsilon) = -\frac{\alpha}{\sqrt{2|C_0|k_0d}} \cdot \varepsilon^{1-\tau} + \min\{O(\varepsilon^{2-2\tau}), O(\varepsilon)\}$$

and

$$q(k; \kappa^0, d, \varepsilon) = -\frac{\alpha}{\sqrt{2|C_0|k_0d}} \cdot \varepsilon^{1-\tau} + \min\{O(\varepsilon^{2-2\tau}), O(\varepsilon)\}$$

when m is odd and even, respectively.

Proof. The lemma follows by noting that $\frac{\cot k}{k} + \frac{1}{k \sin k} = 0$ and $\frac{\cot k}{k} - \frac{1}{k \sin k} = 0$ when $k = m\pi$ for odd and even m , respectively.

From the discussions in section 2, the field enhancement will occur for small p or q . By Lemma 5.1 and the same calculations as in section 4.3, the following proposition holds for the amplitude of the enhanced field when the resonant frequencies are close to the Rayleigh cutoff frequencies.

PROPOSITION 5.2. If the resonant frequency $m\pi$ is $\varepsilon^{2\tau}$ -close to a cutoff frequency k^0 , then the amplitude of the near-field wave is

$$O(\varepsilon^{\tau-1}) \quad \text{and} \quad O(\varepsilon^\tau \ln \varepsilon)$$

inside the slits $S_\varepsilon^{(0),int}$ and on the slit apertures Γ_ε^\pm , respectively.

Therefore, in contrast to the case when the resonant frequency is away from the Rayleigh cutoff frequencies, for which the field is enhanced by an order of $O(\varepsilon^{-1})$, the field enhancement becomes weaker if the resonant frequency is close to one of the Rayleigh cutoff frequencies. In addition, from Lemma 5.1, it is observed that the wavefield at the resonance frequency has a phase difference of π for $\delta > 0$ and $\delta < 0$. \square

6. A discussion on embedded eigenvalues. As discussed in section 4, for each $\kappa \in (-\pi/d, \pi/d]$, there exist real eigenvalues k_m such that $k_m < |\kappa|$ and is below the light line. The corresponding eigenmodes are surface bound states that are confined near the periodic structure. In addition, the dispersion curve $k_m(\kappa)$ is continuous (cf. Lemma 4.5), and such surface bound states are robust in the sense that they persist if κ is perturbed. For the periodic structure, there may exist real eigenvalues that satisfy $k > |\kappa|$ and lie above the light line. Such eigenvalues are embedded in the continuous spectrum, and they coincide with the intersection point of the complex dispersion relation for the quasi-modes [28, 32, 29] and the real (κ, k) -plane. Especially, the corresponding eigenmodes are not robust with respect to a perturbation of κ . The dissolution of embedded eigenvalues in the continuous spectrum is the mechanism behind transmission anomaly and field enhancement for the periodic slab structure (for instance, Fano resonance) when it is illuminated by a plane wave. We refer to [29] and references therein for detailed discussions.

For the periodic structure considered in this paper, unfortunately, such embedded eigenvalues do not exist for the scattering operator $A(\kappa, d, \varepsilon)$ when the domain of the operator is restricted to the space of quasi-periodic functions such that

$$u_\varepsilon(x_1 + d, x_2) = e^{i\kappa d} u_\varepsilon(x_1, x_2)$$

or, more precisely, the function space

$$H_{\kappa, d}^1(\Omega_\varepsilon) := \left\{ u : u \in H^1(\Omega_\varepsilon), u(x_1 + d, x_2) = e^{i\kappa d} u(x_1, x_2) \right\}.$$

Indeed, as discussed in section 4, the associated dispersion relation is determined by the roots of the functions $p(k; \kappa, d, \varepsilon)$ or $q(k; \kappa, d, \varepsilon)$, and its asymptotic expansion is given by (4.1). Moreover, as stated in section 4.2, $\text{Im } \gamma(m\pi, \kappa, d)$ holds as long as $m\pi > |\kappa|$. Therefore, k_m will not be a real eigenvalue if it lies above the light line or when $k_m > |\kappa|$.

However, if we view the periodic structure with a period of $2d$ instead of d , and seek for quasi-periodic solutions in $H_{\kappa, 2d}^1(\Omega_\varepsilon)$ such that

$$u_\varepsilon(x_1 + 2d, x_2) = e^{i\kappa 2d} u_\varepsilon(x_1, x_2),$$

then embedded eigenvalues may exist for the corresponding scattering operator $A(\kappa, 2d, \varepsilon)$. In what follows, we explore the embedded eigenvalues of the operator $A(\kappa, 2d, \varepsilon)$, which follows the construction in [13].

Let us start with an elementary observation. For each fixed $\kappa \in (-\pi/d, \pi/d]$, we define $\hat{\kappa}$ to be one of the two numbers, $\kappa + \pi/d$ or $\kappa - \pi/d$, for which $\hat{\kappa} \in (-\pi/d, \pi/d]$. The Hilbert space $H_{\hat{\kappa}, d}^1(\Omega_\varepsilon)$ is then well defined, and it is clear that $H_{\hat{\kappa}, d}^1(\Omega_\varepsilon) \subset H_{\kappa, 2d}^1(\Omega_\varepsilon)$.

LEMMA 6.1. *The Hilbert space $H_{\kappa, 2d}^1(\Omega_\varepsilon)$ is the orthogonal sum of the two subspaces $H_{\kappa, d}^1(\Omega_\varepsilon)$ and $H_{\hat{\kappa}, d}^1(\Omega_\varepsilon)$.*

Proof. It is clear that $H_{\kappa, d}^1(\Omega_\varepsilon)$ and $H_{\hat{\kappa}, d}^1(\Omega_\varepsilon)$ are subspaces of $H_{\kappa, 2d}^1(\Omega_\varepsilon)$ and they are orthogonal to each other. We only need to show that any functions in $H_{\kappa, 2d}^1(\Omega_\varepsilon)$ can be written as the sum of two functions in $H_{\kappa, d}^1(\Omega_\varepsilon)$ and $H_{\hat{\kappa}, d}^1(\Omega_\varepsilon)$, respectively. Without loss of generality, let us take $\hat{\kappa} = \kappa + \pi/d$.

Let $f \in H_{\kappa, 2d}^1(\Omega_\varepsilon)$. Then f adopts the following representation:

$$f(x_1, x_2) = \sum_n f_n(x_2) e^{i\kappa x_1 + i \frac{n\pi}{d} x_1}$$

when the f_n 's are expansion coefficients. By rearranging the terms in the above series, we see that

$$f = \sum_n f_{2n}(x_2) e^{i\kappa x + i\frac{2n\pi}{d}x_1} + \sum_n f_{2n+1}(x_2) e^{i\hat{\kappa}x + i\frac{2n\pi}{d}x_1}.$$

It is clear that the former belongs to $H_{\kappa,d}^1(\Omega_\varepsilon)$ and the latter belongs to $H_{\hat{\kappa},d}^1(\Omega_\varepsilon)$. This completes the proof of the lemma. \square

As a consequence of the above lemma, we see that the embedded eigenvalues for the operator $A(\kappa, 2d, \varepsilon)$ of the $2d$ -periodic scattering problem are either associated with the eigenvalues of the operator $A(\kappa, d, \varepsilon)$ or those of the operator $A(\hat{\kappa}, d, \varepsilon)$. Equivalently, these correspond to the roots of the functions $p(k, \kappa, d, \varepsilon)$ and $q(k, \kappa, d, \varepsilon)$, or the functions $p(k, \hat{\kappa}, d, \varepsilon)$ and $q(k, \hat{\kappa}, d, \varepsilon)$. We now give a concrete example of such embedded eigenvalues. From Remark 4.2, it is known that if

$$k_1(\pi/d, d, \varepsilon) < \pi/d,$$

then $k_1(\pi/d, d, \varepsilon)$ is an eigenvalue of the operator $A(\hat{\kappa}, d, \varepsilon)$ for $\hat{\kappa} = \pi/d$, and the corresponding surface bound state \tilde{u}_ε satisfies

$$\tilde{u}_\varepsilon(x_1 + d, x_2) = e^{i\pi} \tilde{u}_\varepsilon(x_1, x_2).$$

Therefore, it follows

$$\tilde{u}_\varepsilon(x_1 + 2d, x_2) = \tilde{u}_\varepsilon(x_1, x_2).$$

This shows that $k = k_1(\pi/d, d, \varepsilon)$ is also an eigenvalue of the operator $A(\kappa, 2d, \varepsilon)$ for $\kappa = 0$, and it is an embedded eigenvalue that satisfies $k_1(\pi/d, d, \varepsilon) > |\kappa| = 0$. We may draw a similar conclusion for those (k, κ) near $(\pi/d, 0)$.

In general, if we view the periodic structure with a period of md , where m is an arbitrary positive integer, and seek for the quasi-periodic solutions in $H_{\kappa,md}^1(\Omega_\varepsilon)$ such that

$$u_\varepsilon(x_1 + md, x_2) = e^{i\kappa md} u_\varepsilon(x_1, x_2),$$

a discussion similar to the $2d$ -period scattering problem would show the existence of embedded eigenvalues for the corresponding operator $A(\kappa, md, \varepsilon)$. However, all these embedded eigenvalues are constructed for the d -period scattering problem, and consequently they are robust. Transmission anomaly and field enhancement would not necessarily occur when the periodic slab structure is illuminated by a plane wave with a frequency given by an embedded eigenvalue. Since, as stated at the beginning of this section, transmission anomalies such as the Fano resonant phenomenon occur when the eigenvalue is dissolved into complex-valued resonance, and the corresponding surface bound state is not robust with respect to a perturbation of κ . More precisely, let k_0 be a simple embedded eigenvalue of the operator $A(\kappa_0, md, \varepsilon)$, then a sufficient condition for Fano resonance to occur is the following (cf. [28, 32, 29]): there exist a small neighborhood $U \in \mathbf{C}^2$ of (k_0, κ_0) such that $(k, \kappa) = (k_0, \kappa_0)$ is the unique point in $U \cap \mathbf{R}^2$ satisfying the dispersion relation

$$p(k, \hat{\kappa}, d, \varepsilon) = 0 \quad \text{or} \quad q(k, \hat{\kappa}, d, \varepsilon) = 0.$$

That is, $(k, \kappa) = (k_0, \kappa_0)$ is an isolated pair in the real space \mathbf{R}^2 . However, from previous analysis, it is clear that this condition is not satisfied for embedded eigenvalues constructed for the d -period scattering problem.

Appendix A. Asymptotic expansions of $G_\varepsilon^i(X, Y)$ and $\tilde{G}_\varepsilon^i(X, Y)$. Recall that

$$(A.1) \quad G_\varepsilon^i(X, Y) = \frac{1}{\varepsilon} \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} c_{mn} \alpha_{mn} \right) \cos(m\pi X) \cos(m\pi Y).$$

Let $C_m = \sum_{n=0}^{\infty} c_{mn} \alpha_{mn}$. Then from the representation of elementary functions by series, it can be shown that

$$C_0(k) = \sum_{n=1}^{\infty} \frac{2}{k^2 - (n\pi)^2} + \frac{1}{k^2} = \frac{\cot k}{k},$$

$$\begin{aligned} C_m(k, \varepsilon) &= \sum_{n=1}^{\infty} \frac{4}{k^2 - (m\pi/\varepsilon)^2 - (n\pi)^2} + \frac{2}{k^2 - (m\pi/\varepsilon)^2} \\ &= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2 - k^2}} \coth \left(\sqrt{(m\pi/\varepsilon)^2 - k^2} \right) \\ &= -\frac{2\varepsilon}{m\pi} - \frac{k^2 \varepsilon^3}{m^3 \pi^3} + O \left(\frac{\varepsilon^5}{m^5} \right), \quad m \geq 1. \end{aligned}$$

Substituting into (A.1) yields the desired expansion for $G_\varepsilon^i(X, Y)$ given as follows:

$$\begin{aligned} &G_\varepsilon^i(X, Y) \\ &= \frac{1}{\varepsilon} \left\{ C_0(k) - \sum_{m \geq 1} \frac{2\varepsilon}{\pi m} \cos(m\pi X) \cos(m\pi Y) - \sum_{m \geq 1} \frac{k^2 \varepsilon^3}{m^3 \pi^3} \cos(m\pi X) \cos(m\pi Y) \right. \\ &\quad \left. + O \left(\sum_{m \geq 1} \frac{\varepsilon^5}{m^5} \right) \right\} \\ &= \frac{\cot k}{k\varepsilon} + \left(-\frac{2}{\pi} \right) \left[-\ln 2 - \frac{1}{2} \ln \left(\left| \sin \left(\frac{\pi(X+Y)}{2} \right) \right| \right) - \frac{1}{2} \ln \left(\left| \sin \left(\frac{\pi(X-Y)}{2} \right) \right| \right) \right] \\ &\quad + O(k^2 \varepsilon^2). \end{aligned}$$

On the other hand,

$$\tilde{G}_\varepsilon^i(X, Y) = \frac{1}{\varepsilon} \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} (-1)^n c_{mn} \alpha_{mn} \right) \cos(m\pi X) \cos(m\pi Y).$$

Let $\tilde{C}_m = \sum_{n=0}^{\infty} (-1)^n c_{mn} \alpha_{mn}$. Again, from the following representation of elementary functions as series,

$$\tilde{C}_0(k) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{k^2 - (n\pi)^2} + \frac{1}{k^2} = \frac{1}{k \sin k},$$

$$\begin{aligned} \tilde{C}_m(k, \varepsilon) &= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{k^2 - (m\pi/\varepsilon)^2 - (n\pi)^2} + \frac{2}{(m\pi/\varepsilon)^2} \\ &= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2 - k^2} \sinh \left(\sqrt{(m\pi/\varepsilon)^2 - k^2} \right)} \\ &= O \left(\frac{\varepsilon}{m\pi} e^{-m\pi/\varepsilon} \right), \quad m \geq 1, \end{aligned}$$

we obtain

$$\tilde{G}_\varepsilon^i(X, Y) = \frac{1}{(k \sin k)\varepsilon} + O\left(e^{-1/\varepsilon}\right).$$

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