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RESONANCES OF A FINITE ONE-DIMENSIONAL PHOTONIC CRYSTAL WITH A DEFECT*

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Abstract. This paper is concerned with the scattering resonances of a layered dielectric medium of finite extent. The photonic structure consists of a finite number of periodic layers and some defect embedded in the interior. It is proved that there exist resonances that are close to the point spectrum of an infinite layered structure. Moreover, the distance between such near bound-state resonances and the point spectrum decays exponentially as a function of the number of the periodic layers. A simple numerical method is also presented to calculate the near bound-state resonances accurately.

Key words. photonic crystal, scattering resonances, Helmholtz equation

AMS subject classifications. 35Q60, 35P30

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1. Introduction.

We consider a layered dielectric medium of finite extent, where the transverse magnetic polarized electromagnetic wave propagates perpendicular to the layers (yz plane). The Maxwell equations that model the electromagnetic wave propagation are reduced to the following scalar wave equation for the magnetic field:

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial u}{\partial x}\right) = 0, \\ &u(x,0) = u_0, \quad \frac{\partial u}{\partial t}(x,0) = u_1, \end{aligned} \tag{1.1}$$

where c is the wave speed. Denote the left and right boundaries of the photonic structure by $x = -a$ and a, respectively. The relative permittivity $\varepsilon = 1$ outside the interval $(-a, a)$ and is periodic inside the interval with some defects near the origin $x = 0$ (Figure 1). The region $(-a, a)$ is also known as a cavity. In general the energy leaks from the cavity due to the scattering loss to the surrounding medium.

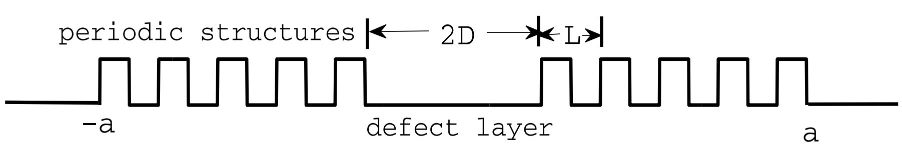

Fig. 1. $\varepsilon(x)$ *of the photonic crystal with defect.*

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Such energy leakage is closely related to scattering resonances associated with the cavity. Considering the time harmonic magnetic field $u(x,t) = \psi(x)e^{-i\omega t}$, the resonances are solutions of the following nonlinear eigenvalue problem:

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\varepsilon}\dfrac{d\psi}{dx}\right) = k^2\psi, \\ \dfrac{1}{\varepsilon}\dfrac{d\psi}{dx}(-a) + ik\psi(-a) = 0, \\ \dfrac{1}{\varepsilon}\dfrac{d\psi}{dx}(a) - ik\psi(a) = 0, \end{cases} \tag{1.2}$$

where $\omega = ck$. The radiation boundary conditions are imposed by noting the fact that an outgoing wave takes the form $\psi = A^+e^{ikx}$ when $x > a$ ($\psi = A^-e^{-ikx}$ when $x < -a$, respectively), and the magnetic field ψ and the electric field $\frac{1}{i\omega\varepsilon}\frac{d\psi}{dx}$ are continuous across the interface $x = a$ ($x = -a$, respectively). In fact, the eigenvalue problem (1.2) attains a sequence of complex resonances k_j with imaginary part $\Im k_j < 0$ and the corresponding quasi modes $\psi_j(x)$, which are locally integrable. Furthermore, the solution of the wave equation (1.1) can be approximated by the modes $e^{-ick_jt}\psi_j(x)$. More precisely, for any $K > 0$, there exist constants $\epsilon(a,K)$, $\tau(a,K)$ such that

$$\left\| u(\cdot,t) - \sum_{\Im k_j > -K} c_j e^{-ick_jt}\psi_j \right\|_{L^2(-a,a)} = O(e^{-K(1+\epsilon)t}), \quad t \geq \tau. \tag{1.3}$$

We refer the reader to [21] for more details of such an approximation. Hence, the energy leakage is controlled by the resonances of (1.2), particularly the eigenvalue k_{j_0} with the smallest magnitude in the negative imaginary part, since asymptotically the solution $u(x,t)$ decreases in the form of $c_0e^{-ick_{j_0}t}\psi_{j_0}(x)$ with respect to time. There has been much effort to design photonic crystals with k_{j_0} that has small imaginary part (or high Q-factors), with an aim to achieve low leakage for the cavity structures; see, for example, [1, 9, 11, 22] and references therein. Such a photonic cavity has found applications in important areas of physics and engineering, such as electron-photon interactions [13], nonlinear optics [20], and quantum information processing [17].

In this paper, we are interested in studying the resonances that are close to the point spectrum σ_p of the infinite layered structure. This main result is stated in Theorem 4.2. In particular, we prove that, for each k_p belonging to the point spectrum σ_p, there exists some resonance k close to k_p. Moreover, the distance $|k - k_p|$ decays exponentially as a function of the number of periodic layers, N. A natural numerical method is also presented for calculating the near bound-state resonance k accurately and efficiently.

It should be mentioned that similar results have also been obtained for a related Schrödinger operator with potential that is a low-energy well surrounded by a thick barrier. In our recent work [4], it is shown that the near bound-state resonance converges to the point spectrum of the associated operator with an infinite-extent potential. Furthermore, the size of the negative imaginary part is exponentially small in the barrier thickness. The explicit solution technique is adopted for the Schrödinger equation in [4]. For the photonic crystal considered here, however, no explicit solution is available. We apply the propagation matrix method, which is discussed in detail in section 2. We also refer the readers to [10, 18] and references therein for the study of resonances and wave propagation for periodic structures without defects.

We focus on the symmetric structure in this paper by assuming that $\varepsilon(x) = \varepsilon(-x)$. The medium is periodic and disturbed with a single defect near the origin. In particular, each period cell consists of two homogeneous layers (Figure 1). In terms of permittivity, let D represent half of the length of the defect layer, and L denote the length of each periodic cell; then the permittivity values inside the cavity $(-a, a)$ can be expressed as follows:

$$\begin{aligned} \varepsilon(x) &= \varepsilon_d, \quad x \in (0, D) \quad \text{(defect)}; \\ \varepsilon(x + D + nL) &= \varepsilon(x + D), \quad x \in (0, L), \quad n = 1, 2, 3, \ldots, N \quad \text{(periodic medium)}; \\ \varepsilon(x + D) &= \varepsilon_1, \quad x \in (0, L_1); \\ \varepsilon(x + D) &= \varepsilon_2, \quad x \in (L_1, L) \quad \text{(two layers in each period)}. \end{aligned}$$

Here we have implicitly defined $a = D + NL$. The quasi modes for the eigenvalue problem (1.2) are even or odd for such structures due to the symmetry property, and we restrict our attention here to the even modes by considering the following equivalent eigenvalue problem:

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\varepsilon}\dfrac{d\psi}{dx}\right) = k^2\psi, \\ \psi(0) = 1, \quad \dfrac{d\psi(0)}{dx} = 0, \\ \dfrac{1}{\varepsilon}\dfrac{d\psi}{dx}(a) - ik\psi(a) = 0. \end{cases} \tag{1.4}$$

The analysis can be carried out for the odd quasi modes in a similar fashion. For simplicity, here and henceforth the differential operator $-\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{d}{dx}\right)$ is denoted by H_N.

The rest of the paper is organized as follows. Section 2 introduces the propagation matrix method. In particular, we carry out the sensitivity analysis on the eigenvalues and eigenvectors of the propagation matrix. Section 3 recalls briefly the spectrum for the structure with infinite-period cells. The main result and its proof are given in section 4. We present a simple method for calculating the near bound-state resonance in section 5 and conclude with some discussions in section 6.

2. Propagation matrix. We study the eigenvalue problem by applying the propagation matrix method, which is convenient for converting the Cauchy problem for the wave equation of the photonic structure to a matrix recurrence equation on the boundaries of layers. In this section, we introduce the propagation matrix briefly and derive some results useful for later sections.

Introduce the vector wave function

$$\Psi(x; k) = \begin{bmatrix} \psi(x; k) \\ \dfrac{1}{\varepsilon(x)}\dfrac{d\psi(x; k)}{dx} \end{bmatrix}, \tag{2.1}$$

where $\psi(x; k)$ is the solution of the ordinary differential equation (ODE)

$$\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{d\psi}{dx}\right) + k^2\psi = 0.$$

Define the matrix

$$T(x; k) = \begin{bmatrix} 0 & \varepsilon(x) \\ -k^2 & 0 \end{bmatrix};$$

then $\Psi(x;k)$ is the solution of the following ODE system:

$$\frac{d}{dx}\Psi(x;k) = T(x;k)\Psi(x;k). \tag{2.2}$$

The propagation of the wave field over each layer with constant permittivity ε can be written as

$$\Psi(x;k) = P(x - x_0;\epsilon,k)\Psi(x_0;k) \quad \text{for } x > x_0,$$

where the propagation matrix

$$P(l;\varepsilon,k) = \begin{bmatrix} \cos(k\sqrt{\varepsilon}\,l) & \dfrac{\sqrt{\varepsilon}}{k}\sin(k\sqrt{\varepsilon}\,l) \\ -\dfrac{k}{\sqrt{\varepsilon}}\sin(k\sqrt{\varepsilon}\,l) & \cos(k\sqrt{\varepsilon}\,l) \end{bmatrix}. \tag{2.3}$$

In addition, the propagation matrix $P(l;\varepsilon,k)$ is unimodular in the sense that $\det[P(l;\varepsilon,k)] = 1$.

Now the propagation matrix over the defect layer is

$$P_D(k) = P(D;\varepsilon_d,k), \tag{2.4}$$

and the propagation matrix $P_L(k)$ for each period of the photonic crystal is the product of the associated propagation matrix over each constant layer. More precisely,

$$P_L(k) = P(L_2;\varepsilon_2,k)P(L_1;\varepsilon_1,k), \tag{2.5}$$

where $L_2 = L - L_1$. For completeness, we express $P_L(k)$ explicitly as follows:

$$P_L(k) = \begin{bmatrix} \cos\mu_1\cos\mu_2 - \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\sin\mu_1\sin\mu_2 & \dfrac{1}{k}\left(\sqrt{\varepsilon_1}\sin\mu_1\cos\mu_2 + \sqrt{\varepsilon_2}\cos\mu_1\sin\mu_2\right) \\ -k\left(\dfrac{1}{\sqrt{\varepsilon_1}}\sin\mu_1\cos\mu_2 + \dfrac{1}{\sqrt{\varepsilon_2}}\cos\mu_1\sin\mu_2\right) & \cos\mu_1\cos\mu_2 - \sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}}\sin\mu_1\sin\mu_2 \end{bmatrix}, \tag{2.6}$$

where $\mu_i = k\sqrt{\varepsilon_i}L_i$ $(i = 1,2)$.

We first examine analyticity of the eigenvalues and the associated eigenvectors for the propagation matrix $P_L(k)$. For completeness, a sketch of the proof is also given in the following; readers are referred to [12, 14] for an alternative proof. Define a function $F(\lambda,k) = \det(\lambda I - P_L(k))$; then two eigenvalues $\lambda_1(k)$ and $\lambda_2(k)$ of $P_L(k)$ are roots of

$$F(\lambda,k) = 0.$$

Let $k_0 \in \mathbb{C}\backslash\{0\}$ such that $P_L(k_0)$ has two distinct eigenvalues $\lambda_{1,0}$ and $\lambda_{2,0}$. Note that $F(\lambda,k)$ is an analytic function in the neighborhood of $(\lambda_{1,0},k_0) \in \mathbb{C}^2$. In addition, at $(\lambda_{1,0},k_0)$, using the fact that $\det(P_L(k)) \equiv 1$, we have

$$\frac{\partial F}{\partial \lambda}(\lambda_{1,0},k_0) = 2\lambda_{1,0} - \mathrm{Trace}(P_L(k_0)) = \lambda_{1,0} - \lambda_{2,0} \neq 0,$$

since two eigenvalues $\lambda_{1,0}$ and $\lambda_{2,0}$ are distinct. Then it follows from the analytic implicit function theorem that $F(\lambda,k) = 0$ has a unique analytic solution, which we denote as eigenvalue $\lambda_1(k)$, in the neighborhood of k_0, and its value equals $\lambda_{1,0}$ at k_0 [15]. Similarly, we denote eigenvalue $\lambda_2(k)$ as the analytic solution of $F(\lambda,k) = 0$ in the neighborhood of k_0 that satisfies $\lambda_2(k_0) = \lambda_{2,0}$.

In the neighborhood of k_0, the corresponding eigenvectors for two distinct eigenvalues $\lambda_1(k)$ and $\lambda_2(k)$ can be expressed explicitly as follows, where the left eigenvectors

$$V_1(k) = \begin{bmatrix} p_{21}^L(k) \\ \lambda_1(k) - p_{11}^L(k) \end{bmatrix}, \qquad V_2(k) = \begin{bmatrix} p_{21}^L(k) \\ \lambda_2(k) - p_{11}^L(k) \end{bmatrix}, \tag{2.7}$$

and the right eigenvectors

$$U_1(k) = \begin{bmatrix} \lambda_1(k) - p_{22}^L(k) \\ p_{21}^L(k) \end{bmatrix}, \qquad U_2(k) = \begin{bmatrix} \lambda_2(k) - p_{22}^L(k) \\ p_{21}^L(k) \end{bmatrix}, \tag{2.8}$$

such that

$$V_1^T P_L = \lambda_1 V_1^T, \quad V_2^T P_L = \lambda_2 V_2^T,$$
$$P_L U_1 = \lambda_1 U_1, \quad P_L U_2 = \lambda_2 U_2.$$

Here p_{ij}^L is an entry of the matrix $P_L(k)$. Furthermore, we normalized the left eigenvectors in such a way that

$$\overline{V}_i^T(k) U_j(k) = \delta_{ij}$$

by letting

$$\overline{V}_1(k) = \frac{1}{p_{21}^L(2\lambda_1 - (p_{11}^L + p_{22}^L))} V_1(k) \quad \text{and} \quad \overline{V}_2(k) = \frac{1}{p_{21}^L(2\lambda_2 - (p_{11}^L + p_{22}^L))} V_2(k). \tag{2.9}$$

The eigenvectors $V_1(k)$, $V_2(k)$, $U_1(k)$, $U_2(k)$ are also analytic in the neighborhood of k_0 since $\lambda_1(k)$, $\lambda_2(k)$, and $p_{ij}^L(k)$ are analytic. We summarize the above discussion in the following lemma.

Lemma 2.1. *Let $k_0 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_{1,0}$ and $\lambda_{2,0}$ are two distinct eigenvalues of the propagation matrix $P_L(k_0)$. Then the eigenvalues $\lambda_1(k)$, $\lambda_2(k)$ that satisfy $\lambda_1(k_0) = \lambda_{1,0}$, $\lambda_2(k_0) = \lambda_{2,0}$, and the corresponding eigenvectors $V_1(k)$, $V_2(k)$, $U_1(k)$, $U_2(k)$ of $P_L(k)$ defined in* (2.7)–(2.8)*, are analytic in the neighborhood of k_0.*

The sensitivity of the right eigenvectors to the parameter k is stated in the following lemma.

Lemma 2.2. *Let $k \in \mathbb{C}\backslash\{0\}$ such that $\lambda_1(k)$ and $\lambda_2(k)$ are two distinct eigenvalues of $P_L(k)$. The corresponding left and right eigenvectors $\overline{V}_1(k)$ and $\overline{V}_2(k)$, $U_1(k)$, $U_2(k)$ are defined by* (2.9) *and* (2.8)*, respectively. Define the matrix $U(k) = [U_1(k), U_2(k)]$; then the derivative $U'(k)$ exists. Furthermore, there exists a 2×2 complex matrix Q such that $U'(k) = U(k)Q$, where the off-diagonal entry of Q,*

$$q_{ij} = \frac{1}{\lambda_j - \lambda_i} \overline{V}_i^T P_L{}' U_j, \qquad i \neq j,\ i,j = 1,2.$$

Proof. The existence of $U'(k)$ follows directly from the analyticity of $U_1(k)$ and $U_2(k)$ in Lemma 2.1. Since the eigenvectors $U_1(k)$ and $U_2(k)$ are linearly independent for two distinct eigenvalues $\lambda_1(k)$ and $\lambda_2(k)$, then there exists a matrix Q such that $U'(k) = U(k)Q$. Now we proceed to calculate its entries q_{ij}.

Let $V(k) = [\overline{V}_1(k), \overline{V}_2(k)]$ and

$$\Lambda(k) = \begin{bmatrix} \lambda_1(k) & 0 \\ 0 & \lambda_2(k) \end{bmatrix}.$$

We have

$$P_L U = U\Lambda \quad \text{and} \quad V^T P_L = \Lambda V^T.$$

Taking the derivative of the first equation and permuting with the left eigenvectors V^T, it follows that

$$V^T P_L' U + V^T P_L U' = V^T U' \Lambda + V^T U \Lambda'.$$

By noting that $V^T U = I$ and $U' = UQ$, we have

$$V^T P_L' U + \Lambda Q = Q\Lambda + \Lambda',$$

i.e.,

$$V^T P_L' U = Q\Lambda - \Lambda Q + \Lambda'. \tag{2.10}$$

Consequently, for $i \neq j$,

$$q_{ij} = \frac{1}{\lambda_j - \lambda_i} \overline{V}_i^T {P_L}' U_j$$

follows by a direct comparison of the corresponding entry on both sides of (2.10). ∎

3. Spectrum of the photonic structure with infinite periods. In this section, we recall briefly the spectrum for the photonic structure with infinite periods, i.e., $N = \infty$. The associated operator H_∞ for the eigenvalue problem,

$$-\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{d\psi}{dx}\right) = k^2\psi, \quad -\infty < x < \infty,$$

is an unbounded self-adjoint operator in $L^2(\mathbb{R})$. The spectrum consists of a set of continuous spectrum and a discrete spectrum on the real line [5].

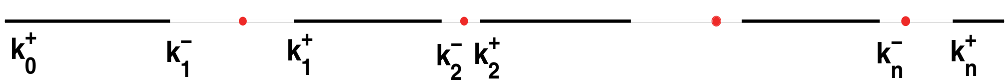

FIG. 2. *Spectrum of H_∞.*

The continuous spectrum $\sigma_c = \cup_{n=1}^{\infty}[k_{n-1}^+, k_n^-]$, where $k_{n-1}^+ < k_n^- < k_n^+$ (see Figure 2), and each set of the continuous spectrum is separated by some gap (k_n^-, k_n^+). It follows from the stability of the essential spectrum by the Weyl theorem that the continuous spectrum σ_c coincides with the entire spectrum of the same periodic structure without the defect layer [6, 19]. Moreover, by the Floquet–Bloch theory [7, 19], the continuous spectrum

$$\sigma_c = \bigcup_{0 \le \tau \le 2\pi} \sigma(H^L(\tau)),$$

where $\sigma(H^L(\tau))$ is the set of eigenvalues for the subproblem defined in one periodic cell:

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\varepsilon_p}\dfrac{d\psi}{dx}\right) = k^2\psi, \quad 0 < x < L, \\ \psi(L) = e^{i\tau}\psi(0), \quad \dfrac{1}{\varepsilon_2}\dfrac{d\psi(L)}{dx} = e^{i\tau}\dfrac{1}{\varepsilon_1}\dfrac{d\psi(0)}{dx}. \end{cases} \tag{3.1}$$

Note that ε_p is the permittivity inside one periodic cell.

The point spectrum $\sigma_p = \cup_{j=1} k_{p_j}$ is induced by the defect layer. It consists of a set of discrete numbers, where each k_{p_j} is located in some gap (k_n^-, k_n^+) (Figure 2). The associated eigenmodes are localized functions that decay exponentially from the defect.

In fact, the spectrum is closely related to the eigenvalues of the propagation matrix $P_L(k)$ over one periodic cell, which is discussed in the following.

Note that here k is real; hence $P_L(k)$ is a 2×2 real matrix, and either the eigenvalues λ_1 and λ_2 of $P_L(k)$ are conjugate pairs or both are real-valued. Furthermore, since $\det(P_L(k)) = 1$, λ_1 and λ_2 may be classified as follows:

(i) $|\lambda_1(k)| > 1 > |\lambda_2(k)|$. In this case λ_1 and λ_2 are real, and $|\mathrm{Trace}(P_L(k))| > 2$. Let $\psi(x;k)$ be the solution of (1.4). Denote the corresponding eigenvectors of λ_1 and λ_2 for $P_L(k)$ by U_1 and U_2, respectively; then the wave field vector $\Psi(D;k)$ at the edge of the defect can be written as a linear combination of two linearly independent eigenvectors:

$$\Psi(D;k) = \alpha_1 U_1 + \alpha_2 U_2.$$

Therefore, the wave field at $x = D + nL$ is

$$\Psi(x;k) = \alpha_1 \lambda_1^n U_1 + \alpha_2 \lambda_2^n U_2. \tag{3.2}$$

The boundedness condition $|\psi(x;k)| < \infty$ is satisfied only if $\alpha_1 = 0$, i.e., when k is some value such that the vector $\Psi(D;k)$ is parallel to the eigenvector U_2. In this case the associated eigenmode $\psi(x;k)$ decays exponentially as $x \to \infty$ with rate $|\lambda_2|^n$.

(ii) $|\lambda_1(k)| = |\lambda_2(k)| = 1$. Since the entries P_L are real-valued, the eigenvalues occur in conjugate pairs:

$$\lambda_1 = e^{i\tau} \quad \text{and} \quad \lambda_2 = e^{-i\tau}, \quad \tau \in [0, 2\pi]. \tag{3.3}$$

Note that for the subproblem (3.1) in one periodic cell, the wave field at the edge of the periodic cell is $P_L(k)\Psi(0;k)$, where $\Psi(0;k)$ is the wave field at $x = 0$. According to Floquet–Bloch theory discussed above, for each $\tau \in [0, 2\pi]$, k belongs to $\sigma(H^L(\tau))$ if and only if $e^{i\tau}$ is the eigenvalue of the propagation matrix $P_L(k)$. Therefore, it follows from (3.3) that the equation $|\lambda_1(k)| = |\lambda_2(k)| = 1$ determines the whole continuous spectrum σ_c, which consists of an infinite set of bounded intervals. For each $k \in \sigma_c$, the associated modes are propagating waves, and τ is also known as the Bloch phase.

In terms of the trace of the propagation matrix $P_L(k)$, the equation for the continuous spectrum reads

$$\mathrm{Trace}(P_L(k)) = 2\cos(\tau), \quad 0 \le \tau \le 2\pi.$$

The continuous spectrum is separated by gaps described by

$$|\mathrm{Trace}(P_L(k))| > 2,$$

with the corresponding eigenvalues for $P_L(k)$ satisfying $|\lambda_1(k)| > 1 > |\lambda_2(k)|$. From the discussion in (i), the isolated point spectrum σ_p with the associated eigenmodes decaying exponentially as $x \to \infty$ lies in those gaps.

We summarize the above discussion for the spectrum of H_∞ in the following proposition and refer the reader to [5] for more details.

PROPOSITION 3.1. *If k belongs to the continuous spectrum σ_c of H_∞, then the eigenvalues of the corresponding propagation matrix $P_L(k_p)$ are conjugate pairs with $|\lambda_1(k)| = |\lambda_2(k)| = 1$. If $k \in \sigma_p$ with the associated eigenmode decaying exponentially away from the defect, then the eigenvalues of $P_L(k)$ satisfy $|\lambda_1(k)| > 1 > |\lambda_2(k)|$. Moreover, the wave field $\Psi(D;k)$ at the edge of the defect layer is parallel to the eigenvector $U_2(k)$ corresponding to the eigenvalue $\lambda_2(k)$.*

4. Near bound-state resonances.

4.1. Main results. We now study the resonances of (1.4). In particular, here we focus on the resonances that are near the point spectrum of the infinite-period structure (near bound-state resonances). In fact, all the resonances of (1.4) lie below the real axis, which is illustrated in the following lemma.

LEMMA 4.1. *Let $k \in \mathbb{C}\backslash\{0\}$ be a resonance of* (1.4)*; then k has negative imaginary part.*

Proof. Multiply the differential equation in (1.4) by $\bar{\psi}$ and integrate by parts; it follows that

$$\int_0^a -\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{d\psi}{dx}\right)\bar{\psi} - k^2|\psi|^2\,dx = \int_0^a \frac{1}{\varepsilon}\left|\frac{d\psi}{dx}\right|^2 - k^2|\psi|^2\,dx - \frac{1}{\varepsilon}\frac{d\psi(a)}{dx}\bar{\psi}(a) = 0.$$

Applying the boundary condition at $x = a$, we find

$$\int_0^a \frac{1}{\varepsilon}\left|\frac{d\psi}{dx}\right|^2 - k^2|\psi|^2\,dx - ik|\psi(a)|^2 = 0. \tag{4.1}$$

Let $k = \Re k + i\Im k$. First let us consider the case when the real part $\Re k \neq 0$. By a simple calculation, the imaginary part of the left-hand side of (4.1) is

$$-\Re k\left(2\Im k\int_0^a |\psi|^2\,dx + |\psi(a)|^2\right).$$

If $\Im k \geq 0$, then $\psi(a) = 0$. Noting the boundary condition at $x = a$, we see $\frac{d\psi(a)}{dx} = 0$. It follows that $\psi \equiv 0$ in $(0,a)$. Therefore, $\Im k < 0$.

Now if the real part $\Re k = 0$, the left-hand side of (4.1) is

$$\int_0^a \frac{1}{\varepsilon}\left|\frac{d\psi}{dx}\right|^2 + (\Im k)^2|\psi|^2\,dx + \Im k|\psi(a)|^2.$$

If $\Im k > 0$, then a similar argument shows that $\psi \equiv 0$ in $(0,a)$. The proof is complete. ∎

The main result regarding the near bound-state resonance is stated in the following theorem.

THEOREM 4.2. *Let k_p belong to the point spectrum of σ_p of H_∞. Then there exists an integer N_0 such that for any $N \geq N_0$ there is resonance k of* (1.4) *in the neighborhood of k_p. Further, the following estimate holds:*

$$|\,k - k_p| \leq Me^{-\beta(k)N},$$

where M is a positive constant independent of N and $\beta(k)$ is a function defined in the neighborhood of k_p with $\beta(k) > 0$.

From Theorem 4.2 and the approximation of the wave function (1.3), it is seen that the energy of the corresponding mode leaks from the cavity to the surrounding medium at a rate of $e^{-c(e^{-\beta N}t)}$. On the other hand, Theorem 4.2 leads to a natural numerical method for calculating the resonances that are near the point spectrum, a method which is presented in section 5.

To validate the conclusion of Theorem 4.2, we consider a photonic structure with half of the defect length $D = 3$; the length of two layers in one period is $L_1 = 2$ and $L_2 = 1$, respectively. The permittivity of each layer is given by $\varepsilon_D = 2$, $\varepsilon_1 = 4$, and $\varepsilon_2 = 1$. The smallest eigenvalue in the point spectrum is calculated by Newton's method with value $k_p = 0.629980209391856$, and the resonance for the photonic structure with different periods near k_p is given in Table 1. It can be seen that the difference between k and k_p decays exponentially as a function of N.

Table 1
Near bound-state resonances (first column), and the difference between the resonances and the point spectrum for different periods (second column).

N	k	$k - k_p$
2	$0.620523719872968 - 0.023447369057171i$	$-9.456489519 \times 10^{-3} - 2.344736906 \times 10^{-2}i$
4	$0.628776070324641 - 0.004027608192325i$	$-1.204139067 \times 10^{-3} - 4.027608192 \times 10^{-3}i$
8	$0.629942017730269 - 0.000146219552859i$	$-3.819166159 \times 10^{-5} - 1.462195529 \times 10^{-4}i$
16	$0.629980156982703 - 0.000000202962748i$	$-5.247756196 \times 10^{-8} - 2.029627483 \times 10^{-7}i$

4.2. Proof of main result. Note that the spectrum for the infinite structure is symmetric with respect to the origin; we need only to consider positive k_p in the proof.

Lemma 4.3. *Letting Ψ_0 be a vector in $\mathbb{C}^2$, $k \in \mathbb{C}\backslash\{0\}$, then*

$$P_L'(k)\Psi_0 = P_L(k)\int_0^L P^{-1}(x;k)O(k)P(x;k)\Psi_0 dx, \tag{4.2}$$

where

$$O(k) = \begin{bmatrix} 0 & 0 \\ -2k & 0 \end{bmatrix}, \tag{4.3}$$

and $P(x;k)$ is the propagation matrix inside one periodic cell given by

$$P(x;k) = \begin{cases} P(x;\varepsilon_1,k), & x \in [0, L_1], \\ P(x - L_1;\varepsilon_2,k)P(L_1;\varepsilon_1,k), & x \in [L_1, L]. \end{cases} \tag{4.4}$$

Proof. Let

$$\Psi(L;k) = P_L(k)\Psi_0. \tag{4.5}$$

From (2.1)–(2.2) we see that $\Psi(L;k)$ defined by (4.5) is the solution of the following ODE system:

$$\begin{cases} \dfrac{d}{dx}\Psi(x;k) = T(x;k)\Psi(x;k), \\ \\ \Psi(0;k) = \Psi_0 \end{cases} \tag{4.6}$$

at $x = L$ with $\varepsilon(x)$ given in one periodic cell.

Taking the derivative for both sides of (4.5) with respect to k, then

$$\Psi'(L;k) = P_L'(k)\Psi_0. \tag{4.7}$$

On the other hand, from (4.6) and the continuous dependence of the solution to the ODE on the parameter [8], $\Psi'(L;k)$ is the solution of the following ODE system:

$$\begin{cases} \dfrac{d}{dx}\Psi'(x;k) = T(x;k)\Psi'(x;k) + O(k)\Psi(x;k), \\ \Psi'(0;k) = [0,0]^T \end{cases}$$

at $x = L$. The solution of the above ODE system can be expressed in the form

$$\Psi'(L;k) = P_L(k)\int_0^L P^{-1}(x;k)O(k)\Psi(x;k)\,dx.$$

Hence, by noting that $\Psi(x;k) = P(x;k)\Psi_0$ and combining this with (4.7), we have

$$P_L'(k)\Psi_0 = P_L(k)\int_0^L P^{-1}(x;k)O(k)P(x;k)\Psi_0\,dx. \qquad \square$$

THEOREM 4.4. *Let $k \in \mathbb{C}$ such that $\Re k > 0$ and $\Im k = 0$, and let the eigenvalues of $P_L(k)$ satisfy $|\lambda_1(k)| > 1 > |\lambda_2(k)|$. Let $U_2(k)$ be the eigenvector of $\lambda_2(k)$ defined by* (2.8)*. Then* $\det[U_2(k), U_2'(k)] > 0$.

Proof. From Lemma 2.2, the derivative $U_2'(k)$ exists. Furthermore, there exist complex numbers q_{12} and q_{22} such that

$$U_2' = q_{12}U_1 + q_{22}U_2,$$

where

$$q_{12} = \frac{1}{\lambda_2 - \lambda_1}\overline{V}_1^T P_L{}' U_2.$$

Therefore, by noting that $\det[U_2, U_2] = 0$ and $\det[U_2, U_1] = p_{21}^L(\lambda_2 - \lambda_1)$, we have

$$\det[U_2, U_2'] = q_{12}\det[U_2, U_1] = \frac{1}{\lambda_2 - \lambda_1}\overline{V}_1^T P_L{}' U_2 \det[U_2, U_1] = p_{21}^L \overline{V}_1^T P_L{}' U_2.$$

Since, from (2.9), the normalized left eigenvector

$$\overline{V}_1(k) = \frac{1}{p_{21}^L(2\lambda_1 - (p_{11}^L + p_{22}^L))}V_1(k),$$

by substituting into the above equation, it follows that

$$\det[U_2, U_2'] = \frac{1}{(2\lambda_1 - (p_{11}^L + p_{22}^L))}V_1^T P_L{}' U_2. \tag{4.8}$$

On the other hand, by Lemma 4.3,

$$P_L'(k)U_2 = P_L(k)\int_0^L P^{-1}(x;k)O(k)P(x;k)U_2\,dx, \tag{4.9}$$

with the matrices $O(k)$ and $P(x,k)$ given by (4.3) and (4.4), respectively. Substituting (4.9) into (4.8) yields

$$\det[U_2, U_2'] = \frac{1}{(2\lambda_1 - (p_{11}^L + p_{22}^L))} V_1^T P_L(k) \int_0^L P^{-1}(x;k)O(k)P(x;k)U_2 \, dx$$
$$= \frac{\lambda_1}{(\lambda_1 - \lambda_2)} V_1^T \int_0^L P^{-1}(x;k)O(k)P(x;k)U_2 \, dx,$$

where the second equality follows by noting that $V_1^T P_L = \lambda_1 V_1^T$ and $\lambda_1 + \lambda_2 = p_{11}^L + p_{22}^L$.

From the expressions of the eigenvectors in (2.7), (2.8) and the equality $\lambda_1 + \lambda_2 = p_{11}^L + p_{22}^L$, it is seen that $V_1 = \tilde{O}U_2$, where

$$\tilde{O} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Consequently,

$$\det[U_2, U_2'] = \frac{\lambda_1}{(\lambda_1 - \lambda_2)} \int_0^L U_2^T \, [\, \tilde{O}^T P^{-1}(x;k)O(k)P(x;k) \,] \, U_2 \, dx. \tag{4.10}$$

Since $\det(P(x;k)) \equiv 1$, for each $x \in [0, L]$ we may express $P^{-1}(x;k)$ explicitly as

$$P^{-1}(x;k) = \begin{bmatrix} p_{22}(x;k) & -p_{12}(x;k) \\ -p_{21}(x;k) & p_{11}(x;k) \end{bmatrix},$$

where $p_{ij}(x;k)$ is the entry of the propagation matrix $P(x;k)$. A direct calculation yields

$$\tilde{O}^T P^{-1}(x;k)O(k)P(x;k) = 2k \begin{bmatrix} p_{11}^2(x;k) & p_{11}(x;k)p_{12}(x;k) \\ p_{11}(x;k)p_{12}(x;k) & p_{12}^2(x;k) \end{bmatrix}.$$

We denote the right-hand side of the above formula by $S(x;k)$. Then for each $x \in [0, L]$ it is seen that $S(x;k)$ is a semipositive definite matrix when $\Re k > 0$ and $\Im k = 0$; i.e.,

$$U_2^T \, S(x;k) \, U_2 \geq 0 \qquad \forall x \in [0, L].$$

Moreover,

$$U_2^T \, S(x;k) \, U_2 = 0 \quad \text{if and only if} \quad U_2 \cdot \tilde{p}(x;k) = 0,$$

where the vector $\tilde{p}(x;k) = [p_{11}(x;k), p_{12}(x;k)]^T$.

Note that

$$\tilde{p}(x;k) = [\, \cos(k\sqrt{\varepsilon}x), \sqrt{\varepsilon}/k \sin(k\sqrt{\varepsilon}x) \,]^T$$

for $x \in [0, L_1]$, and U_2 is independent of x. We see there exists at least one point $x_0 \in [0, L_1]$ such that $U_2 \cdot \tilde{p}(x_0, k) \neq 0$; hence

$$U_2^T \, S(x_0;k) \, U_2 > 0.$$

By continuity of the matrix $S(x;k)$ it follows that

$$\int_0^L U_2^T \, [\, \tilde{O}^T P^{-1}(x;k)O(k)P(x;k) \,] \, U_2 \, dx > 0. \tag{4.11}$$

Finally, since k is real and $|\lambda_1(k)| > 1 > |\lambda_2(k)|$, both $\lambda_1(k)$ and $\lambda_2(k)$ are real by noting that $P_L(k)$ is a 2×2 real matrix and $\det(P_L(k)) = 1$. Consequently,

$$\frac{\lambda_1}{(\lambda_1 - \lambda_2)} > 0, \tag{4.12}$$

by using the fact that $\lambda_1(k)$ and $\lambda_2(k)$ are real, and $|\lambda_1(k)| > 1 > |\lambda_2(k)|$. We conclude that $\det[U_2, U_2'] > 0$, by combining (4.10), (4.11), and (4.12). ∎

Remark. Following a proof similar to that of Theorem 4.4, it can be shown that for $k \in \mathbb{C}$ with $\Re k > 0$ and $\Im k = 0$, and the eigenvalues of $P_L(k)$ satisfying $|\lambda_1(k)| > 1 > |\lambda_2(k)|$, the determinant $\det[U_1, U_1'] < 0$, where U_1 is the eigenvector of λ_1 defined by (2.8).

If k_p belongs to the point spectrum σ_p of the infinite-period structure, then from Proposition 3.1 two eigenvalues of $P_L(k_p)$ are distinct: $|\lambda_1(k_p)| > 1 > |\lambda_2(k_p)|$. Hence, regarding the eigenvector U_2, we have the following corollary.

COROLLARY 4.5. *If $k_p > 0$ belongs to the point spectrum σ_p of the infinite-period structure, then $\det[U_2(k_p), U_2'(k_p)] > 0$, where U_2 is the eigenvector defined by* (2.8).

Let $k \in \mathbb{C}$ be in the neighborhood of some point spectrum $k_p > 0$ such that two eigenvalues of $P_L(k)$ are distinct. $U_2(k)$ defined by (2.8) is the eigenvector corresponding to the eigenvalue with smaller modulus. Let $\Psi(D; k)$ be the field vector at the edge of the defect for the eigenvalue problem (1.4). We define the matrix

$$\Phi(k) = [\Psi(D; k), U_2(k)]. \tag{4.13}$$

THEOREM 4.6. *The determinant $\det \Phi(k)$ is analytic in the neighborhood of some point spectrum $k_p > 0$. Moreover, $\det \Phi(k_p) = 0$, and there exists a constant $\gamma > 0$ such that $|(\det \Phi(k_p))'| \geq \gamma$.*

Proof. By the propagation matrix, the field

$$\Psi(D; k) = P_D(k) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where the matrix P_D is given by (2.4). Note that $P_D(k)$ is analytic in the neighborhood of k_p; hence by Lemma 2.1 it is seen that $\det(\Phi(k))$ is analytic. On the other hand, from Proposition 3.1, $\det \Phi(k_p) = 0$ follows directly from the fact that the vector $\Psi(D; k)$ is parallel to the eigenvector $U_2(k)$ at $k = k_p$.

To prove the last assertion, we see, by direct calculations, that

$$(\det \Phi(k_p))' = \det[\Psi'(D; k_p), U_2(k_p)] + \det[\Psi(D; k_p), U_2'(k_p)].$$

Since the vector $\Psi(D; k)$ is parallel to the eigenvector $U_2(k)$ at the point spectrum $k = k_p$, there exists a nonzero real constant α_2 such that

$$\Psi(D; k_p) = \alpha_2 U_2(k_p).$$

Therefore,

$$(\det \Phi(k_p))' = \frac{1}{\alpha_2} \det[\Psi'(D; k_p), \Psi(D; k_p)] + \alpha_2 \det[U_2(k_p), U_2'(k_p)]. \tag{4.14}$$

By a direct calculation, it follows that

$$\det[\Psi'(D; k_p), \Psi(D; k_p)] = \frac{1}{2\sqrt{\varepsilon_D}}(\mu_D + \sin(\mu_D)) > 0, \tag{4.15}$$

since $\mu_D = 2\sqrt{\varepsilon_D}\, k_p D > 0$. On the other hand, by Corollary 4.5, we have

$$\det[U_2(k_p), U_2'(k_p)] > 0. \tag{4.16}$$

Hence the proof is complete by combining (4.14), (4.15), and (4.16). ∎

Now we are ready to prove Theorem 4.2.

Proof. Since $|\lambda_1(k_p)| > 1 > |\lambda_2(k_p)|$, there exists a neighborhood of the k_p such that the eigenvalues $\lambda_1(k)$ and $\lambda_2(k)$ of $P_L(k)$ are distinct:

$$|\lambda_1(k)| > 1 > |\lambda_2(k)|\,.$$

Hence the field vector $\Psi(D;k)$ at the edge of the defect can be written as a linear combination of the two linearly independent eigenvectors $U_1(k)$ and $U_2(k)$:

$$\Psi(D;k) = \alpha_1 U_1(k) + \alpha_2 U_2(k), \tag{4.17}$$

where

$$U_1(k) = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} \quad \text{and} \quad U_2(k) = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

are given by (2.8). By applying the propagation matrix, the field at the end of the structure $x = D + NL$ is expressed in the following form:

$$\Psi(D+NL;k) = \alpha_1 \lambda_1^N U_1(k) + \alpha_2 \lambda_2^N U_2(k). \tag{4.18}$$

Enforcing the boundary condition at $x = D + NL$, it follows that

$$\alpha_1 \lambda_1^N U_1(k) + \alpha_2 \lambda_2^N U_2(k) = \begin{bmatrix} C \\ ikC \end{bmatrix}$$

for some constant C or, equivalently,

$$\lambda_1^N (iku_{11} - u_{21})\alpha_1 + \lambda_2^N (iku_{12} - u_{22})\alpha_2 = 0. \tag{4.19}$$

On the other hand, from (4.17), the unknowns α_1 and α_2 may be expressed in terms of $\Psi(D;k)$, $U_1(k)$, and $U_2(k)$:

$$\alpha_1 = \frac{\det[\Psi(D;k), U_2(k)]}{\det[U_1(k), U_2(k)]} \quad \text{and} \quad \alpha_2 = \frac{\det[U_1(k), \Psi(D;k)]}{\det[U_1(k), U_2(k)]}.$$

Substituting into (4.19), and noting that $\det[U_1, U_2] \neq 0$, we arrive at

$$\lambda_1^N (iku_{11} - u_{21}) \det[\Psi(D;k), U_2(k)] + \lambda_2^N (iku_{12} - u_{22}) \det[U_1(k), \Psi(D;k)] \;=\; 0, \tag{4.20}$$

which is the equation of resonance.

Define

$$F(k) = (iku_{11} - u_{21}) \det[\Psi(D;k), U_2(k)] \tag{4.21}$$

and

$$G(k) = (iku_{12} - u_{22}) \det[U_1(k), \Psi(D;k)]; \tag{4.22}$$

then the resonance condition (4.20) is recast as

$$F(k) = -\left(\frac{\lambda_2}{\lambda_1}\right)^N G(k). \tag{4.23}$$

Here $F(k)$ and $G(k)$ are analytic in the neighborhood of the point spectrum k_p. Furthermore, at $k = k_p$, by noting that $\det[\Psi(D; k_p), U_2(k_p)] = 0$ in Theorem 4.6, we see

$$F'(k_p) = (ik_p u_{11}(k_p) - u_{21}(k_p))(\det[\Psi(D; k), U_2(k)])'|_{k=k_p}.$$

It is clear that $ik_p u_{11}(k_p) - u_{21}(k_p) \neq 0$ since both $u_{11}(k_p)$ and $u_{21}(k_p)$ are real from the expression (2.8) for real k_p. On the other hand, $(\det[\Psi(D; k), U_2(k)])' \neq 0$ at $k = k_p$ by Theorem 4.6, and we conclude that

$$|F'(k_p)| > 0. \tag{4.24}$$

By Taylor's theorem [2], there exists an analytic function $\tilde{F}(k)$ such that

$$F(k) = F(k_p) + \tilde{F}(k)(k - k_p) \tag{4.25}$$

and

$$\tilde{F}(k_p) = F'(k_p).$$

In addition, from (4.24) it follows that there exists a constant $\gamma > 0$ such that

$$\left|\tilde{F}(k)\right| \geq \gamma \quad \text{in the neighborhood of } k_p.$$

Now, noting that $F(k_p) = 0$ by using the fact $\det[\Psi(D; k_p), U_2(k_p)] = 0$, we simplify (4.23) as

$$k = k_p - \left(\frac{\lambda_2}{\lambda_1}\right)^N \frac{G(k)}{\tilde{F}(k)}. \tag{4.26}$$

On the other hand, $|\lambda_1(k)| > 1 > |\lambda_2(k)|$, and thus there exists a function $\beta(k) > 0$ such that

$$\left|\frac{\lambda_2}{\lambda_1}\right| = e^{-\beta(k)} \quad \text{in the neighborhood of } k_p.$$

Since $|G(k)| \leq \tilde{M}$ for some constant $\tilde{M}$ by analyticity, now the estimate

$$|\, k - k_p| \leq M e^{-\beta(k)N}$$

holds with the constant $M = \frac{\tilde{M}}{\gamma}$.

Denote the map $k_p - \left(\frac{\lambda_2}{\lambda_1}\right)^N \frac{G(k)}{\tilde{F}(k)}$ on the right-hand side of (4.26) by $\mathcal{R}(k)$. The derivative is

$$\mathcal{R}'(k) = N\left(\frac{\lambda_2}{\lambda_1}\right)^{N-1}\left(\frac{\lambda_2}{\lambda_1}\right)'\frac{G(k)}{\tilde{F}(k)} + \left(\frac{\lambda_2}{\lambda_1}\right)^N\left(\frac{G(k)}{\tilde{F}(k)}\right)'.$$

Note again that $|\lambda_1(k)| > 1 > |\lambda_2(k)|$. By choosing N sufficiently large, it is easily seen that $|\mathcal{R}'(k)| < 1$ in the neighborhood of k_p. Hence $\mathcal{R}(k)$ is a contraction map, and the existence of the resonance follows. $\square$

TABLE 2
Resonance k and the associated numerical approximation k_{approx} for different layers.

N	k	k_{approx}
2	$7.389183413 \times 10^{-1} - 4.641019263 \times 10^{-3}i$	$7.389647014 \times 10^{-1} - 4.51969770 \times 10^{-3}i$
4	$7.384993087 \times 10^{-1} - 1.384595010 \times 10^{-4}i$	$7.384994091 \times 10^{-1} - 1.38331515 \times 10^{-4}i$
8	$7.384847323 \times 10^{-1} - 1.295821199 \times 10^{-7}i$	$7.384847323 \times 10^{-1} - 1.295819840 \times 10^{-7}i$
16	$7.384847185 \times 10^{-1} - 1.137082964 \times 10^{-13}i$	$7.384847186 \times 10^{-1} - 1.137081535 \times 10^{-13}i$

5. Numerical approximation of near bound-state resonances. Theorem 4.2 suggests a simple numerical method for calculating the near bound-state resonance. Since $|k - k_p|$ is exponentially small, one can calculate k by a linear approximation of the resonance condition (4.23):

$$k_{approx} \approx k_p - \left(\frac{\lambda_2(k_p)}{\lambda_1(k_p)}\right)^N \frac{G(k_p)}{F'(k_p)}, \tag{5.1}$$

where the analytic functions $F(k)$ and $G(k)$ are defined by (4.21) and (4.22), and $\lambda_1(k)$ and $\lambda_2(k)$ are two eigenvalues of the propagation matrix $P_L(k)$.

The associated quasi mode $\Psi(x)$ is then approximated by applying the propagation matrix. More specifically,

$$\Psi(x) = P(x - nL; k_{approx}) P_L^n(k_{approx}) P_D(k_{approx}) \Psi_0, \qquad nL \le x \le (n+1)L,$$

where

$$P(x;k) = \begin{cases} P(x;\varepsilon_1,k), & x \in [0, L_1], \\ P(x - L_1; \varepsilon_2, k) P(L_1; \varepsilon_1, k), & x \in [L_1, L]. \end{cases}$$

We present an example to show the accuracy of the approximation. Consider a photonic structure with $D = 1.5$; the length of two layers inside one period cell is $L_1 = 2$ and $L_2 = 1$, respectively, and the permittivity of each layer inside the cavity is $\varepsilon_D = 2$, $\varepsilon_1 = 1$, and $\varepsilon_2 = 6$. The resonance k is computed by Newton's method. The comparison between k and its approximation by (5.1) for different layers is summarized in Table 2. The error $\|\psi - \psi_{approx}\|_{L^2(-a,a)}$ of the quasi mode for different numbers of periods is plotted in Figure 3. Figure 4 shows the real and imaginary parts of ψ_{approx} with $N = 4$. It is seen that both the resonance k and its associated quasi mode ψ are approximated accurately. In particular, the errors decay with increasing numbers of periods N.

6. Discussion. We have shown that for a photonic crystal of finite extent with some defect there exists some resonance k near the point spectrum of the infinite-period photonic structure. Moreover, the resonance k converges exponentially fast to the point spectrum k_p with respect to the number of periods N. Based on this observation, a simple method has been proposed to calculate the near bound-state resonances.

A natural interesting question can be raised: what can we say about other resonances of (1.2)? By numerical calculations, for a structure with parameters specified by $D = 1.5$, $L_1 = 2$, $L_2 = 1$, $\varepsilon_D = 2$, $\varepsilon_1 = 1$, and $\varepsilon_2 = 6$, the resonances for different period layers are depicted in Figure 5. It is observed that the density of resonance increases linearly with N. Moreover, it is seen that, as N increases, the non–near bound-state resonances converge to the continuous spectrum of the infinite structure. However, compared to the near bound-state resonances, the convergence rate is much

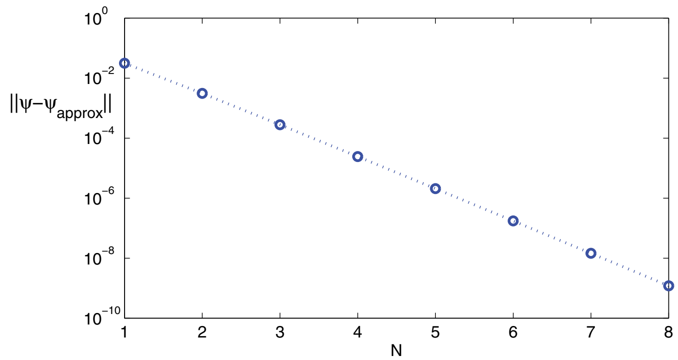

FIG. 3. *Approximation error* $\|\psi - \psi_{approx}\|_{L^2(-a,a)}$ *for different numbers of periods.*

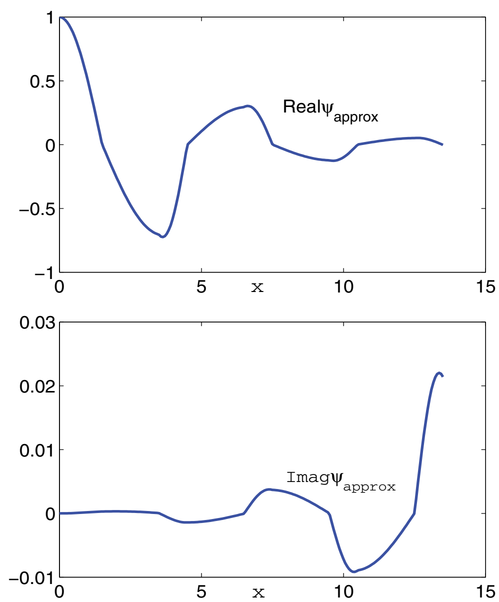

FIG. 4. *Real and imaginary parts of the approximated quasi mode* ψ_{approx} *for* $N = 4$.

slower. In this specific example, the resonances approach the continuous spectrum with a rate of $\frac{1}{N}$. Theoretical validation of the observation is our ongoing work and will be reported elsewhere.

Another direction is the study of resonances for photonic structures with the coexistence of both left- and right-handed materials. It is shown that metamaterial suppresses localization in the distorted layered media [3, 16]. However, its impact on the associated resonances and their relation with eigenvalues of the infinite structure is not yet clear.

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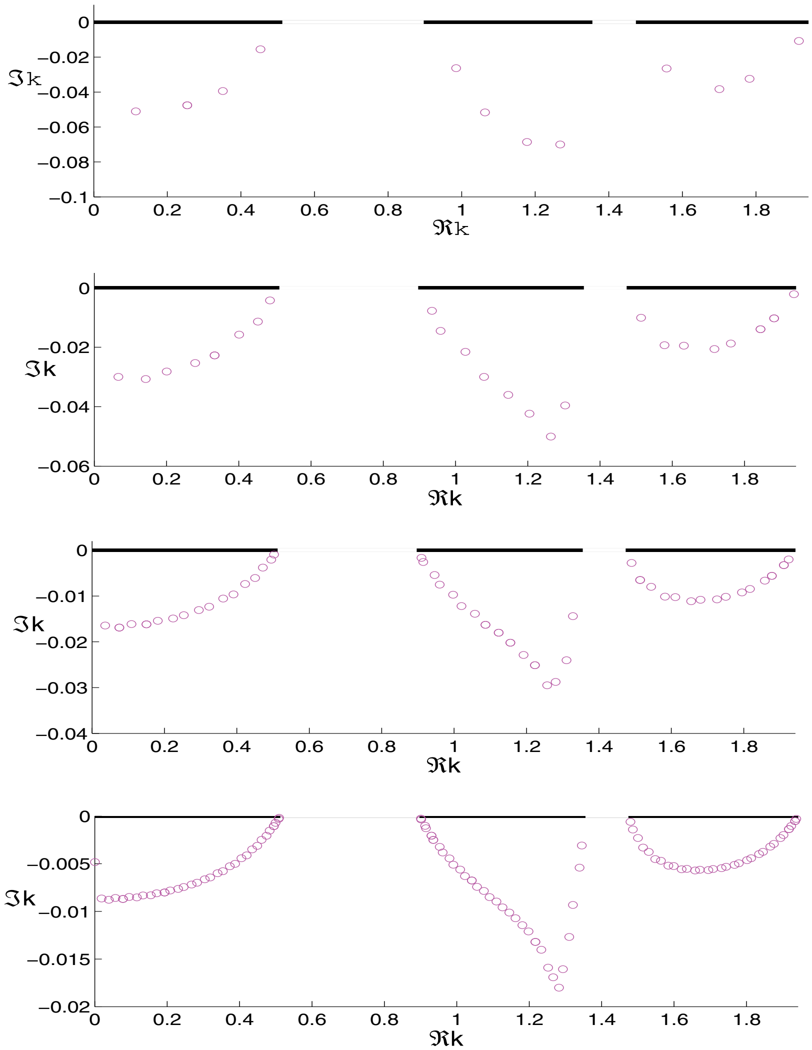

FIG. 5. *Non–near bound-state resonances for $N = 2, 4, 8, 16$. $\Re k$ and $\Im k$ denote the real and imaginary parts of the resonance, respectively.*

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