

An Efficient Numerical Method for Acoustic Wave Scattering in Random Media*

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Abstract. This paper is concerned with developing efficient numerical methods for acoustic wave scattering in random media which can be expressed as random perturbations of homogeneous media. We first analyze the random Helmholtz problem by deriving some wavenumber-explicit solution estimates. We then establish a multimodes representation of the solution as a power series of the perturbation parameter and analyze its finite modes approximations. Based on this multimodes representation, we develop a Monte Carlo interior penalty discontinuous Galerkin (MCIP-DG) method for approximating the mode functions, which are governed by recursively defined nearly deterministic Helmholtz equations. Optimal order error estimates are derived for the method, and an efficient algorithm, which is based on the LU direct solver, is also designed for efficiently implementing the proposed multimodes MCIP-DG method. It is proved that the computational complexity of the whole algorithm is comparable to that of solving one deterministic Helmholtz problem using the LU direct solver. Numerical experiments are provided to validate the theoretical results and to gauge the performance of the proposed numerical method and algorithm.

Key words. Helmholtz equation, random media, Rellich identity, discontinuous Galerkin method, error estimate, LU decomposition, Monte Carlo method

AMS subject classifications. 65N12, 65N15, 65N30

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1. Introduction. Partial differential equations (PDEs) with random coefficients arise naturally in the modeling of many physical phenomena. This is due to the fact that some level of uncertainty is usually involved if one's knowledge of the physical behavior is not complete or when noise is present in the experimental measurements. In recent years, substantial progress has been made in the numerical approximation of such PDEs due to the significant development in computational resources. We refer the reader to [1, 2, 3, 16, 17] and references therein for more details.

In this paper, we consider the propagation of the acoustic wave in a medium where the wave velocity is characterized by a random process. More precisely, we study the approximation of

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the solution to the following Helmholtz problem:

$$(1.1) \quad -\Delta u(\omega, \cdot) - k^2 \alpha(\omega, \cdot)^2 u(\omega, \cdot) = f(\omega, \cdot) \quad \text{in } D,$$

$$(1.2) \quad \partial_\nu u(\omega, \cdot) + ik\alpha(\omega, \cdot)u(\omega, \cdot) = 0 \quad \text{on } \partial D,$$

where k is the wavenumber, and $D \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is a convex bounded polygonal domain with boundary ∂D . ν denotes the unit outward normal on ∂D . Let (Ω, \mathcal{F}, P) be a probability space with sample space Ω , σ -algebra \mathcal{F} , and probability measure P . For each fixed $x \in D$, the refractive index $\alpha(\cdot, x)$ is a real-valued random variable defined over Ω . We assume that the medium is a small random perturbation of a uniform background medium in the sense that

$$(1.3) \quad \alpha(\omega, \cdot) := 1 + \varepsilon\eta(\omega, \cdot).$$

Here ε represents the magnitude of the random fluctuation, and $\eta \in L^2(\Omega, L^\infty(D))$ is some random process satisfying

$$P \left\{ \omega \in \Omega; \|\eta(\omega, \cdot)\|_{L^\infty(D)} \leq 1 \right\} = 1.$$

For notation brevity we only consider the case that η is real-valued. However, we note that the results of this paper are also valid for complex-valued η . On the boundary ∂D , a radiation boundary condition is imposed to absorb incoming waves [6]. Here ν denotes the unit outward normal to ∂D , and $\partial_\nu u$ stands for the normal derivative of u . The boundary value problem (1.1)–(1.2) arises in the modeling of the wave propagation in complex environments, such as composite materials, oil reservoirs, and geological basins [9, 12]. In such instances, it is of practical interest to characterize the uncertainty of the wave energy transport when the medium contains some randomness. In particular, we are interested in the computation of some statistics of the wave field, e.g., the mean value of the solution u .

To solve stochastic (or random) partial differential equations (SPDEs) numerically, the simplest and most natural approach is to use the Monte Carlo method, where a set of independent identically distributed (i.i.d.) solutions are obtained by sampling the PDE coefficients, and the mean of the solution is calculated via a statistical average over all the sampling in the probability space [3]. An alternative is the stochastic Galerkin method, where the SPDE is reduced to a high-dimensional deterministic equation by expanding the random field in the equation using the Karhunen–Loève or Wiener chaos expansions. We refer the reader to [1, 2, 5, 16, 17] for detailed discussions. However, it is known that a brute-force Monte Carlo or stochastic Galerkin method applied directly to the Helmholtz equation with random coefficients is computationally prohibitive even for a moderate wavenumber k , since a large number of degrees of freedom is involved in the spatial discretization. It is apparent that in such cases, the Monte Carlo method requires solving a PDE with many sampled coefficients, while the high-dimensional deterministic equation associated with the stochastic Galerkin method will be too expensive to solve.

In this paper, we propose an efficient numerical method for solving the Helmholtz problem (1.1)–(1.2) when the medium is weakly random defined by (1.3). A multimodes representation of the solution is derived, where each mode is governed by a Helmholtz equation with

deterministic coefficients and a random source. We develop a Monte Carlo interior penalty discontinuous Galerkin (MCIP-DG) method for approximating the mode functions. In particular, we take advantage of the fact that the coefficients of the Helmholtz equation for all the modes are identical, and hence the associated discretized equations share the same constant coefficient matrix. Using this crucial fact, it is observed that an LU direct solver for the discretized equations leads to a tremendous saving in the computational costs, since the LU decomposition matrices can be used repeatedly, and the solutions for all modes and all samples can be obtained in an efficient way by performing simple forward and backward substitutions. Indeed, it turns out that the computational complexity of the proposed algorithm is comparable to that of solving one deterministic Helmholtz problem using the LU direct solver.

The rest of the paper is organized as follow. A wavenumber-explicit estimate for the solution of the random Helmholtz equation is established in section 2. In section 3, we introduce the multimodes expansion of the solution as a power series of ε and analyze the error estimation for its finite-modes approximation. The Monte Carlo interior penalty discontinuous Galerkin method is presented in section 4, where the error estimates for the approximation of each mode function are also obtained. In section 5, a numerical procedure for solving (1.1)–(1.2) is described and its computational complexity is analyzed in detail. In addition, we derive an optimal order error estimate for the proposed procedure. Several numerical experiments are provided in section 6 to demonstrate the efficiency of the method and to validate the theoretical results.

2. PDE analysis.

2.1. Preliminaries. Standard function and space notation is adopted in this paper. $L^2(D)$ is the space of all square-integrable functions on D , and $H^s(D)$ denotes the standard complex-valued Sobolev space. $(\cdot, \cdot)_S$ stands for the inner product in $L^2(S)$ for some subset S of D , and $\langle \cdot, \cdot \rangle_{\partial D}$ is the inner product on the boundary ∂D . C and c denote generic constants which are independent of k and the mesh parameter h . We also define spaces

$$(2.1) \quad H_+^1(D) := \{v \in H^1(D); |\nabla v|_{\partial D} \in L^2(\partial D)\},$$

$$(2.2) \quad V := \{v \in H^1(D); \Delta v \in L^2(D)\}.$$

Here $|\nabla v|_{\partial D}$ denotes the trace of the Euclidean length $|\nabla v|$ on ∂D . Without loss of generality, we assume that the domain $D \subset B_R(0)$. Throughout this paper we also assume that D is a star-shaped domain with respect to the origin in the sense that there exists a positive constant c_0 such that

$$x \cdot \nu \geq c_0 \quad \text{on } \partial D.$$

Let (Ω, \mathcal{F}, P) be a probability space on which all the random variables of this paper are defined. $\mathbb{E}(\cdot)$ denotes the expectation operator. The abbreviation *a.s.* stands for *almost surely*.

As it will be needed in the late sections of the paper, in this section we analyze the boundary value problem for the Helmholtz equation (1.1) with the following slightly more general nonhomogeneous boundary condition:

$$(2.3) \quad \partial_\nu u(\omega, \cdot) + \mathbf{i}k\alpha(\omega, \cdot)u(\omega, \cdot) = g(\omega, \cdot).$$

Definition 2.1. Let $f \in L^2(\Omega, L^2(D))$ and $g \in L^2(\Omega, L^2(\partial D))$. A function $u \in L^2(\Omega, H^1(D))$ is called a weak solution to problem (1.1), (2.3) if it satisfies the following identity:

$$(2.4) \quad \int_{\Omega} a(u, v) dP = \int_{\Omega} ((f, v)_D + \langle g, v \rangle_{\partial D}) dP \quad \forall v \in L^2(\Omega, H^1(D)),$$

where

$$(2.5) \quad a(w, v) := (\nabla w, \nabla v)_D - k^2(\alpha^2 w, v)_D + ik \langle \alpha w, v \rangle_{\partial D}.$$

Remark 2.1. Using (2.7) below, it is easy to show that any solution u of (1.1), (2.3) satisfies $u \in L^2(\Omega, H^1_+(D) \cap V)$.

2.2. Wavenumber-explicit solution estimates. In this subsection we shall derive stability estimates for the solution of problem (1.1), (2.3) which is defined in Definition 2.1. Our focus is to obtain explicit dependence of the stability constants on the wavenumber k ; such wavenumber-explicit stability estimates will play a vital role in our convergence analysis in later sections. We note that wavenumber-explicit stability estimates also play a pivotal role in the development of numerical methods, such as finite element and discontinuous Galerkin methods, for deterministic reduced wave equations (cf. [7, 8]). As a byproduct of the stability estimates, the existence and uniqueness of solutions to problem (1.1), (2.3) can be conveniently established.

Lemma 2.2. Let $u \in L^2(\Omega, H^1(D))$ be a solution of (1.1), (2.3); then for any $\delta_1, \delta_2 > 0$ and $\varepsilon < 1$ there hold

$$(2.6) \quad \mathbb{E}(\|\nabla u\|_{L^2(D)}^2) \leq \left(k^2(1 + \varepsilon)^2 + \delta_1\right) \mathbb{E}(\|u\|_{L^2(D)}^2) + \left(\frac{\delta_1}{2k^2(1 - \varepsilon)^2} + \frac{1}{2\delta_1}\right) \left(\mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{L^2(\partial D)}^2)\right),$$

$$(2.7) \quad \mathbb{E}(\|u\|_{L^2(\partial D)}^2) \leq \frac{\delta_2}{k(1 - \varepsilon)} \mathbb{E}(\|u\|_{L^2(D)}^2) + \frac{1}{\delta_2 k(1 - \varepsilon)} \mathbb{E}(\|f\|_{L^2(D)}^2) + \frac{1}{k^2(1 - \varepsilon)^2} \mathbb{E}(\|g\|_{L^2(\partial D)}^2).$$

Proof. Setting $v = u$ in (2.4) yields

$$\int_{\Omega} a(u, u) dP = \int_{\Omega} ((f, u)_D + \langle g, u \rangle_{\partial D}) dP.$$

Taking the real and imaginary parts and using the definition of $a(\cdot, \cdot)$, we get

$$(2.8) \quad \int_{\Omega} \left(\|\nabla u\|_{L^2(D)}^2 - k^2(1 + \varepsilon\eta)^2 \|u\|_{L^2(D)}^2\right) dP = \operatorname{Re} \int_{\Omega} ((f, u)_D + \langle g, u \rangle_{\partial D}) dP,$$

$$(2.9) \quad k \int_{\Omega} \langle 1 + \varepsilon\eta, |u|^2 \rangle_{\partial D} dP = \operatorname{Im} \int_{\Omega} ((f, u)_D + \langle g, u \rangle_{\partial D}) dP.$$

Applying the Cauchy–Schwarz inequality to (2.9) produces

$$k(1 - \varepsilon)\mathbb{E}(\|u\|_{L^2(\partial D)}^2) \leq \frac{\delta_2}{2}\mathbb{E}(\|u\|_{L^2(D)}^2) + \frac{1}{2\delta_2}\mathbb{E}(\|f\|_{L^2(D)}^2) \\ + \frac{k(1 - \varepsilon)}{2}\mathbb{E}(\|u\|_{L^2(\partial D)}^2) + \frac{1}{2k(1 - \varepsilon)}\mathbb{E}(\|g\|_{L^2(\partial D)}^2).$$

Thus, (2.7) holds. Applying Cauchy–Schwarz to (2.8) yields

$$\mathbb{E}(\|\nabla u\|_{L^2(D)}^2) \leq \left(k^2(1 + \varepsilon)^2 + \frac{\delta_1}{2}\right)\mathbb{E}(\|u\|_{L^2(D)}^2) + \frac{1}{2\delta_1}\mathbb{E}(\|f\|_{L^2(D)}^2) \\ + \frac{\delta_1}{2}\mathbb{E}(\|u\|_{L^2(\partial D)}^2) + \frac{1}{2\delta_1}\mathbb{E}(\|g\|_{L^2(\partial D)}^2).$$

To this one can apply (2.7) with $\delta_2 = k(1 - \varepsilon)$ and obtain (2.6). The proof is complete. ■

Lemma 2.3. *Let $u \in L^2(\Omega, H^2(D))$; then there hold*

$$(2.10) \quad \operatorname{Re} \int_{\Omega} (u, x \cdot \nabla u)_D dP = -\frac{d}{2} \int_{\Omega} \|u\|_{L^2(D)}^2 dP + \frac{1}{2} \int_{\Omega} \langle x \cdot \nu, |u|^2 \rangle_{\partial D} dP,$$

$$(2.11) \quad \operatorname{Re} \int_{\Omega} (\nabla u, \nabla(x \cdot \nabla u))_D dP = \frac{2-d}{2} \int_{\Omega} \|\nabla u\|_{L^2(D)}^2 dP \\ + \frac{1}{2} \int_{\Omega} \langle x \cdot \nu, |\nabla u|^2 \rangle_{\partial D} dP.$$

Proof. Equation (2.10) follows immediately from applying the divergence theorem to

$$\int_{\Omega} (\operatorname{div}(x|u|^2), 1)_D dP$$

and from the fact that $\operatorname{div}(x) = d$. To show (2.11), we first recall the following differential identities [4]:

$$\nabla \cdot (x|\nabla u|^2) = d|\nabla u|^2 + x \cdot \nabla(|\nabla u|^2), \\ x \cdot \nabla(|\nabla u|^2) = 2\operatorname{Re}(\nabla \cdot (\nabla u \overline{(x \cdot \nabla u)}) - \Delta u \overline{(x \cdot \nabla u)}) - 2|\nabla u|^2 \\ = 2\operatorname{Re}(\nabla u \cdot \overline{\nabla(x \cdot \nabla u)}) - 2|\nabla u|^2.$$

Then (2.11) follows from adding the above two identities, integrating the sum over $D \times \Omega$, and applying the divergence theorem on the left-hand side of the resulting equation. ■

Remark 2.2. Equation (2.11) could be called a stochastic Rellich identity for the Laplacian.

We are now ready to state and prove our wavenumber-explicit estimate for solutions of problem (1.1), (2.3) defined in Definition 2.1.

Theorem 2.4. *Let $u \in L^2(\Omega, H^1(D))$ be a solution of (1.1), (2.3), and let R be the smallest number such that $B_R(0)$ contains the domain D . Then there hold the following estimates:*

$$(2.12) \quad \mathbb{E}\left(\|u\|_{L^2(D)}^2 + \|u\|_{L^2(\partial D)}^2 + c_0 \|\nabla u\|_{L^2(\partial D)}^2\right) \leq C_0 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 M(f, g),$$

$$(2.13) \quad \mathbb{E}(\|u\|_{H^1(D)}^2) \leq C_0 \left(1 + \frac{1}{k^2}\right)^2 M(f, g),$$

provided that $\varepsilon(2 + \varepsilon) < \gamma_0 := \min\{1, \frac{13-2d}{2(4d-7)+25kR}\}$, where C_0 is some positive constant independent of k and u , and

$$(2.14) \quad M(f, g) := \mathbb{E}\left(\|f\|_{L^2(D)}^2 + \|g\|_{L^2(\partial D)}^2\right).$$

Moreover, if $g \in L^2(\Omega, H^{\frac{1}{2}}(D))$ and $u \in L^2(\Omega, H^2(D))$, there also holds

$$(2.15) \quad \mathbb{E}(\|u\|_{H^2(D)}^2) \leq C\left(k + \frac{1}{k^2}\right)^2 \mathbb{E}\left(\|f\|_{L^2(D)}^2 + \|g\|_{H^{\frac{1}{2}}(\partial D)}^2\right).$$

Proof. To avoid some technicalities, below we only give a proof for the case $u \in L^2(\Omega, H^2(D))$. For the general case, u needs be replaced by its mollification u_ρ at the beginning of the proof and followed by taking the limit $\rho \rightarrow 0$ after the integration by parts is done.

Setting $v = x \cdot \nabla u$ in (2.4) yields

$$(2.16) \quad \int_{\Omega} \left((\nabla u, \nabla v)_D - k^2(\alpha^2 u, v)_D + \mathbf{i}k \langle \alpha u, v \rangle_{\partial D} \right) dP = \int_{\Omega} \left((f, v)_D + \langle g, v \rangle_{\partial D} \right) dP.$$

Using (2.10) and (2.11) after taking the real part of (2.16) and regrouping, we get

$$\begin{aligned} \frac{dk^2}{2} \int_{\Omega} \|u\|_{L^2(D)}^2 dP &= \int_{\Omega} \left(\frac{d-2}{2} \|\nabla u\|_{L^2(D)}^2 + k^2 \varepsilon \operatorname{Re}(\eta(2 + \varepsilon\eta), v)_D \right) dP \\ &\quad - \int_{\Omega} \left(k \operatorname{Im} \langle (1 + \varepsilon\eta)u, v \rangle_{\partial D} + \frac{1}{2} \langle x \cdot \nu, |\nabla u|^2 \rangle_{\partial D} - \frac{k^2}{2} \langle x \cdot \nu, |u|^2 \rangle_{\partial D} \right) dP \\ &\quad + \int_{\Omega} \left(\operatorname{Re}(f, v)_D + \operatorname{Re}(g, v)_{\partial D} \right) dP. \end{aligned}$$

It then follows from the Schwarz inequality, the “star-shaped” condition, and the facts that $|x| \leq R$ for $x \in D$ and $\|\eta\|_{L^\infty(D)} \leq 1$ a.s. that

$$\begin{aligned} \frac{dk^2}{2} \mathbb{E}(\|u\|_{L^2(D)}^2) &\leq \frac{d-2}{2} \mathbb{E}(\|\nabla u\|_{L^2(D)}^2) + k^2 \varepsilon R(2 + \varepsilon) \left(\frac{1}{2\delta_1} \mathbb{E}(\|u\|_{L^2(D)}^2) \right. \\ &\quad \left. + \frac{\delta_1}{2} \mathbb{E}(\|\nabla u\|_{L^2(D)}^2) \right) + \frac{R}{2\delta_2} \mathbb{E}(\|f\|_{L^2(D)}^2) + \frac{R\delta_2}{2} \mathbb{E}(\|\nabla u\|_{L^2(D)}^2) \\ &\quad + \frac{R}{2\delta_3} \mathbb{E}(\|g\|_{L^2(\partial D)}^2) + \frac{R\delta_3}{2} \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2) \\ &\quad + \frac{kR}{\delta_4} \mathbb{E}(\|u\|_{L^2(\partial D)}^2) + kR\delta_4 \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2) \\ &\quad - \frac{c_0}{2} \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2) + \frac{k^2 R}{2} \mathbb{E}(\|u\|_{L^2(\partial D)}^2). \end{aligned}$$

At this point, we note that $\varepsilon(2 + \varepsilon) \leq 1$ implies $\varepsilon \leq \frac{1}{2}$. Setting $\delta_3 = \frac{c_0}{4R}$ and $\delta_4 = \frac{c_0}{8kR}$, denoting

$\gamma = \varepsilon(2 + \varepsilon)$, and using Lemma 2.2, we can bound the right-hand side as follows:

$$\begin{aligned} \frac{dk^2}{2} \mathbb{E}(\|u\|_{L^2(D)}^2) &\leq \left(\frac{d-2}{2} + \frac{k^2 R \gamma \delta_1}{2} + \frac{R \delta_2}{2} \right) \mathbb{E}(\|\nabla u\|_{L^2(D)}^2) + \frac{k^2 R \gamma}{2 \delta_1} \mathbb{E}(\|u\|_{L^2(D)}^2) \\ &\quad + \left(\frac{8k^2 R^2}{c_0} + \frac{k^2 R}{2} \right) \mathbb{E}(\|u\|_{L^2(\partial D)}^2) - \frac{c_0}{4} \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2) \\ &\quad + \frac{R}{2 \delta_2} \mathbb{E}(\|f\|_{L^2(D)}^2) + \frac{2R^2}{c_0} \mathbb{E}(\|g\|_{L^2(\partial D)}^2) \\ &\leq \left(\frac{d-2}{2} + \frac{k^2 R \gamma \delta_1}{2} + \frac{R \delta_2}{2} \right) (k^2(1 + \gamma) + \delta_5) \mathbb{E}(\|u\|_{L^2(D)}^2) \\ &\quad + \left(\frac{d-2}{2} + \frac{k^2 R \gamma \delta_1}{2} + \frac{R \delta_2}{2} \right) \left(\frac{2\delta_5}{k^2} + \frac{1}{2\delta_5} \right) \left(\mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{L^2(\partial D)}^2) \right) \\ &\quad + \left(\frac{8k^2 R^2}{c_0} + \frac{k^2 R}{2} \right) \left(2\delta_6 \mathbb{E}(\|u\|_{L^2(D)}^2) + \frac{2}{k^2 \delta_6} \mathbb{E}(\|f\|_{L^2(D)}^2) + \frac{4}{k^2} \mathbb{E}(\|g\|_{L^2(\partial D)}^2) \right) \\ &\quad + \frac{k^2 R \gamma}{2 \delta_1} \mathbb{E}(\|u\|_{L^2(D)}^2) + \frac{R}{2 \delta_2} \mathbb{E}(\|f\|_{L^2(D)}^2) + \frac{2R^2}{c_0} \mathbb{E}(\|g\|_{L^2(\partial D)}^2) - \frac{c_0}{4} \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2), \end{aligned}$$

which is equivalent to

$$(2.17) \quad c_1 \mathbb{E}(\|u\|_{L^2(D)}^2) + \frac{c_0}{4} \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2) \leq c_2 \mathbb{E}(\|f\|_{L^2(D)}^2),$$

where

$$\begin{aligned} c_1 &:= k^2 - \frac{d-2}{2} (k^2 \gamma + \delta_5) - \left(\frac{k^2 R \gamma \delta_1}{2} + \frac{R \delta_2}{2} \right) (k^2(1 + \gamma) + \delta_5) \\ &\quad - \left(\frac{16k^2 R^2}{c_0} + k^2 R \right) \delta_6 - \frac{k^2 R \gamma}{2 \delta_1}, \\ c_2 &:= \left(\frac{d-2}{2} + \frac{k^2 R \gamma \delta_1}{2} + \frac{R \delta_2}{2} \right) \left(\frac{2\delta_5}{k^2} + \frac{1}{2\delta_5} \right) \\ &\quad + \left(\frac{32k^2 R^2}{c_0} + 2k^2 R \right) \left(\frac{2}{k^2 \delta_6} + \frac{4}{k^2} \right) + \frac{R}{2\delta_2} + \frac{2R^2}{c_0}. \end{aligned}$$

Let $\delta_1 = \frac{1}{2k}$, $\delta_2 = \frac{1}{4R}$, $\delta_5 = \frac{k^2}{4}$, and $\delta_6 = \frac{1}{4(\frac{16R^2}{c_0} + R)}$; then

$$\begin{aligned} c_1 &= k^2 \left[\frac{27-4d}{32} - \left(\frac{4d-7}{8} + \frac{(21+4\gamma)Rk}{16} \right) \gamma \right], \\ c_2 &= \left(\frac{d-2}{2} + \frac{kR\gamma}{4} + \frac{1}{8} \right) \left(\frac{1}{2} + \frac{1}{8k^2} \right) + 2R^2(1 + c_0) \\ &\quad + 8 \left(\frac{16R^2}{c_0} + R \right) \left(\frac{16R^2}{c_0} + R + 1 \right). \end{aligned}$$

If $\gamma < \gamma_0$, it is easy to check that $c_1 \geq \frac{k^2}{32}$. Thus, (2.17) implies that

$$(2.18) \quad \mathbb{E}(\|u\|_{0,D}^2) + c_0 \mathbb{E}(\|\nabla u\|_{L^2(\partial D)}^2) \leq \frac{C}{k^2} \left(1 + \frac{1}{k^2} \right) \left(\mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{L^2(\partial D)}^2) \right)$$

for some constant $C > 0$ independent of k and u . This then proves (2.12).

By (2.6) with $\delta_1 = k^2$ and (2.18) we get

$$\begin{aligned} \mathbb{E}(\|u\|_{H^1(D)}^2) &= \mathbb{E}(\|u\|_{L^2(D)}^2) + \mathbb{E}(\|\nabla u\|_{L^2(D)}^2) \\ &\leq \frac{C}{k^2} \left(1 + \frac{1}{k^2}\right) \left(\mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{L^2(\partial D)}^2)\right) \\ &\quad + (k^2(1 + \varepsilon)^4 + k^2) \mathbb{E}(\|u\|_{L^2(D)}^2) \\ &\quad + \left(1 + \frac{1}{2k^2}\right) \left(\mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{L^2(\partial D)}^2)\right) \\ &\leq C \left(1 + \frac{1}{k^2}\right)^2 \left(\mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{L^2(\partial D)}^2)\right). \end{aligned}$$

Hence, (2.13) holds.

Finally, it follows from the standard elliptic regularity theory for the Poisson equation and the trace inequality (cf. [11]) that

$$\begin{aligned} \mathbb{E}(\|u\|_{H^2(D)}^2) &\leq C \left(\mathbb{E}(\|k^2 u\|_{L^2(D)}^2) + \mathbb{E}(\|f\|_{L^2(D)}^2) + \mathbb{E}(\|g\|_{H^{\frac{1}{2}}(\partial D)}^2)\right) \\ &\quad + \mathbb{E}(\|ku\|_{H^{\frac{1}{2}}(\partial D)}^2 + \mathbb{E}(\|u\|_{L^2(D)}^2)) \\ &\leq C \mathbb{E} \left(k^4 \|u\|_{L^2(D)}^2 + \|f\|_{L^2(D)}^2 + \|g\|_{H^{\frac{1}{2}}(\partial D)}^2\right) \\ &\quad + C \mathbb{E} \left(k^2 \|\nabla u\|_{L^2(D)}^2 + \|u\|_{L^2(D)}^2\right) \\ &\leq C \left(k + \frac{1}{k^2}\right)^2 \mathbb{E} \left(\|f\|_{L^2(D)}^2 + \|g\|_{H^{\frac{1}{2}}(\partial D)}^2\right). \end{aligned}$$

Hence (2.15) holds. The proof is complete. ■

Remark 2.3. By the definition of γ_0 , we see that $\gamma_0 = O(\frac{1}{kR})$. In practice, this is not a restrictive condition because R is often taken to be proportional to the wave length. Hence, $kR = O(1)$.

As a nontrivial byproduct, the above stability estimates can be used conveniently to establish the existence and uniqueness of solutions to problem (2.4)–(2.5) as defined in Definition 2.1.

Theorem 2.5. *Let $f \in L^2(\Omega, L^2(D))$ and $g \in L^2(\Omega, L^2(\partial D))$. For each fixed pair of positive number k and ε such that $\varepsilon(2 + \varepsilon) < \gamma_0$, there exists a unique solution $u \in L^2(\Omega, H^1_+(D)) \cap V$ to problem (2.4)–(2.5).*

Proof. The proof is based on the well-known Fredholm alternative principle (cf. [11]). First, it is easy to check that the sesquilinear form on the right-hand side of (2.4) satisfies a Gårding inequality on the space $L^2(\Omega, H^1(D))$. Second, to apply the Fredholm alternative principle we need to prove that solutions to the adjoint problem of (2.4)–(2.5) are unique. It is easy to verify that the adjoint problem is associated with the sesquilinear form

$$\hat{a}(w, v) := (\nabla w, \nabla v)_D - k^2(\alpha^2 w, v)_D - ik \langle \alpha w, v \rangle_{\partial D},$$

which differs from $a(\cdot, \cdot)$ only in the sign of the last term. As a result, all the stability estimates for problem (2.4)–(2.5) still hold for its adjoint problem. Since the adjoint problem is a linear

problem (so is problem (2.4)–(2.5)), the stability estimates immediately imply uniqueness. Finally, the Fredholm alternative principle then implies that problem (2.4)–(2.5) has a unique solution $u \in L^2(\Omega, H^1(D))$. The proof is complete. ■

Remark 2.4. The uniqueness of the adjoint problem can also be proved using the classical unique continuation argument (cf. [13]).

3. Multimodes representation of the solution and its finite modes approximations. The first goal of this section is to develop a multimodes representation for the solution to problem (1.1)–(1.2) in terms of powers of the parameter ε . We first postulate such a representation and then prove its validity by establishing some energy estimates for all the mode functions. The second goal of this section is to establish an error estimate for finite modes approximations of the solution. Both the multimodes representation and its finite modes approximations play a pivotal role in our overall solution procedure for solving problem (1.1)–(1.2) as they provide the theoretical foundation for the solution procedure. Throughout this section, we use u^ε to denote the solution to problem (1.1)–(1.2), which is proved in Theorem 2.5.

Without loss of generality, here and henceforth we assume that $k \geq 1$ and $D \subset B_1(0)$. Otherwise, the problem can be rescaled to this regime by a suitable change of variable. We note that the normalization $D \subset B_1(0)$ implies that $R = 1$. We start by postulating that the solution u^ε has the multimodes expansion

$$(3.1) \quad u^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_n,$$

whose validity will be justified later.

Substituting the above expansion into the Helmholtz equation (1.1) and matching the coefficients of ε^n order terms for $n = 0, 1, 2, \dots$, we obtain

$$(3.2) \quad u_{-1} := 0,$$

$$(3.3) \quad -\Delta u_0 - k^2 u_0 = f,$$

$$(3.4) \quad -\Delta u_n - k^2 u_n = 2k^2 \eta u_{n-1} + k^2 \eta^2 u_{n-2} \quad \text{for } n \geq 1.$$

Similarly, the boundary condition (1.2) translates to each mode function u_n as follows:

$$(3.5) \quad \partial_\nu u_n + \mathbf{i}k u_n = -\mathbf{i}k \eta u_{n-1} \quad \text{for } n \geq 0.$$

An interesting feature of the above multimodes expansion is that all the mode functions satisfy the same type (nearly deterministic) Helmholtz equation and the same boundary condition. The only difference is that the Helmholtz equations have different right-hand side source terms (all of them except one are random variables), and each pair of consecutive mode functions supplies the source term for the Helmholtz equation satisfied by the next mode function. This important feature will be utilized in section 5 to construct our overall numerical methodology for solving problem (1.1)–(1.2).

Next, we address the existence and uniqueness of each mode function u_n .

Theorem 3.1. *Let $f \in L^2(\Omega, L^2(D))$. Then for each $n \geq 0$, there exists a unique solution $u_n \in L^2(\Omega, H^1(D))$ (understood in the sense of Definition 2.1) to problem (3.3), (3.5) for*

$n = 0$ and problem (3.4), (3.5) for $n \geq 1$. Moreover, for $n \geq 0$, u_n satisfies

$$(3.6) \quad \mathbb{E} \left(\|u_n\|_{L^2(D)}^2 + \|u_n\|_{L^2(\partial D)}^2 + c_0 \|\nabla u_n\|_{L^2(\partial D)}^2 \right) \leq \left(\frac{1}{k} + \frac{1}{k^2} \right)^2 C(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2),$$

$$(3.7) \quad \mathbb{E}(\|u_n\|_{H^1(D)}^2) \leq \left(1 + \frac{1}{k^2} \right)^2 C(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2),$$

where

$$(3.8) \quad C(0, k) := C_0, \quad C(n, k) := 4^{2n-1} C_0^{n+1} (1+k)^{2n} \quad \text{for } n \geq 1.$$

Moreover, if $u_n \in L^2(\Omega, H^2(D))$, there also holds

$$(3.9) \quad \mathbb{E}(\|u_n\|_{H^2(D)}^2) \leq \frac{1}{\bar{c}_0} \left(k + \frac{1}{k^2} \right)^2 C(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2),$$

where $\bar{c}_0 := \min\{1, kc_0\}$.

Proof. For each $n \geq 0$, the PDE problem associated with u_n is the same type of Helmholtz problem as the original problem (1.1)–(1.2) (with $\varepsilon = 0$ on the left-hand side of the PDE). Hence, all a priori estimates of Theorem 2.4 hold for each u_n (with its respective right-hand side source function). First, we have

$$(3.10) \quad \mathbb{E} \left(\|u_0\|_{L^2(D)}^2 + \|u_0\|_{L^2(\partial D)}^2 + c_0 \|\nabla u_0\|_{L^2(\partial D)}^2 \right) \leq C_0 \left(\frac{1}{k} + \frac{1}{k^2} \right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2),$$

$$(3.11) \quad \mathbb{E}(\|u_0\|_{H^1(D)}^2) \leq C_0 \left(1 + \frac{1}{k^2} \right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2).$$

Thus, (3.6) and (3.7) hold for $n = 0$.

Next, we use induction to prove that (3.6) and (3.7) hold for all $n > 0$. Assume that (3.6) and (3.7) hold for all $0 \leq n \leq \ell - 1$; then

$$\begin{aligned} & \mathbb{E} \left(\|u_\ell\|_{L^2(D)}^2 + \|u_\ell\|_{L^2(\partial D)}^2 + c_0 \|\nabla u_\ell\|_{L^2(\partial D)}^2 \right) \\ & \leq 2C_0 \left(\frac{1}{k} + \frac{1}{k^2} \right)^2 \mathbb{E} \left(\|2k^2 \eta u_{\ell-1}\|_{L^2(D)}^2 + \bar{\delta}_{1\ell} \|k^2 \eta^2 u_{\ell-2}\|_{L^2(D)}^2 + \|k \eta u_{\ell-1}\|_{L^2(\partial D)}^2 \right) \\ & \leq 2C_0 \left(\frac{1}{k} + \frac{1}{k^2} \right)^2 (1+k)^2 \left(4C(\ell-1, k) + C(\ell-2, k) \right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ & \leq \left(\frac{1}{k} + \frac{1}{k^2} \right)^2 8C_0 (1+k)^2 C(\ell-1, k) \left(1 + \frac{C(\ell-2, k)}{C(\ell-1, k)} \right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ & \leq \left(\frac{1}{k} + \frac{1}{k^2} \right)^2 C(\ell, k) \mathbb{E}(\|f\|_{L^2(D)}^2), \end{aligned}$$

where $\bar{\delta}_{1\ell} = 1 - \delta_{1\ell}$ and $\delta_{1\ell}$ denotes the Kronecker delta, and we have used the fact that $k \geq 1$ and

$$8C_0 (1+k)^2 C(\ell-1, k) \left(1 + \frac{C(\ell-2, k)}{C(\ell-1, k)} \right) \leq C(\ell, k).$$

Similarly, we have

$$\begin{aligned} \mathbb{E}(\|u_\ell\|_{H^1(\partial D)}^2) &\leq 2C_0 \left(1 + \frac{1}{k^2}\right)^2 \mathbb{E} \left(\|2k^2\eta u_{\ell-1}\|_{L^2(D)}^2 + \bar{\delta}_{1\ell} \|k^2\eta^2 u_{\ell-2}\|_{L^2(D)}^2 \right. \\ &\quad \left. + \|k\eta u_{\ell-1}\|_{L^2(\partial D)}^2 \right) \\ &\leq 2C_0 \left(1 + \frac{1}{k^2}\right)^2 (1+k)^2 \left(4C(\ell-1, k) + C(\ell-2, k)\right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ &\leq \left(1 + \frac{1}{k^2}\right)^2 C(\ell, k) \mathbb{E}(\|f\|_{L^2(D)}^2). \end{aligned}$$

Hence, (3.6) and (3.7) hold for $n = \ell$. So the induction argument is complete.

We now use (3.6) and the elliptic theory for Poisson problems directly to verify estimate (3.9):

$$\begin{aligned} \mathbb{E}(\|u_n\|_{H^2(\partial D)}^2) &\leq 2C_0 \left(k + \frac{1}{k^2}\right)^2 \mathbb{E} \left(\|2k^2\eta u_{n-1}\|_{L^2(D)}^2 + \|k^2\eta^2 u_{n-2}\|_{L^2(D)}^2 \right. \\ &\quad \left. + \|k\eta u_{n-1}\|_{H^{\frac{1}{2}}(\partial D)}^2 \right) \\ &\leq 2C_0 \left(k + \frac{1}{k^2}\right)^2 k^2 \mathbb{E} \left(4\|u_{n-1}\|_{L^2(D)}^2 + \|u_{n-2}\|_{L^2(D)}^2 \right. \\ &\quad \left. + \frac{c_0}{kc_0} \|\nabla u_{n-1}\|_{L^2(\partial D)}^2 \right) \\ &\leq \frac{2}{c_0} C_0 \left(k + \frac{1}{k^2}\right)^2 (1+k)^2 \left(4C(n-1, k) + C(n-2, k)\right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ &\leq \frac{1}{c_0} \left(k + \frac{1}{k^2}\right)^2 C(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2). \end{aligned}$$

Hence, (3.9) holds for all $n \geq 0$.

With a priori estimates (3.6) and (3.7) in hand, the proof of existence and uniqueness of each u_n follows verbatim the proof of Theorem 2.5, which we leave to the interested reader to verify. The proof is complete. ■

Now we are ready to justify the multimodes representation (3.1) for the solution u^ε of problem (1.1)–(1.2).

Theorem 3.2. *Let $\{u_n\}$ be the same as in Theorem 3.1. Then (3.1) is valid in $L^2(\Omega, H^1(D))$, provided that $\sigma := 4\varepsilon C_0^{\frac{1}{2}}(1+k) < 1$.*

Proof. The proof consists of two parts: (i) the infinite series on the right-hand side of (3.1) converges in $L^2(\Omega, H^1(D))$; (ii) the limit coincides with the solution u^ε . To prove (i), we define the partial sum

$$(3.12) \quad U_N^\varepsilon := \sum_{n=0}^{N-1} \varepsilon^n u_n.$$

Then for any fixed positive integer p we have

$$U_{N+p}^\varepsilon - U_N^\varepsilon = \sum_{n=N}^{N+p-1} \varepsilon^n u_n.$$

It follows from the Schwarz inequality and (3.6) that for $j = 0, 1$

$$\begin{aligned} \mathbb{E}(\|U_{N+p}^\varepsilon - U_N^\varepsilon\|_{H^j(D)}^2) &\leq p \sum_{n=N}^{N+p-1} \varepsilon^{2n} \mathbb{E}(\|u_n\|_{H^j(D)}^2) \\ &\leq p \left(k^{j-1} + \frac{1}{k^2}\right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2) \sum_{n=N}^{N+p-1} \varepsilon^{2n} C(n, k) \\ &\leq C_0 p \left(k^{j-1} + \frac{1}{k^2}\right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2) \sum_{n=N}^{N+p-1} \sigma^{2n} \\ &\leq C_0 p \left(k^{j-1} + \frac{1}{k^2}\right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2) \cdot \frac{\sigma^{2N}(1 - \sigma^{2p})}{1 - \sigma^2}. \end{aligned}$$

Thus, if $\sigma < 1$, we have

$$\lim_{N \rightarrow \infty} \mathbb{E}(\|U_{N+p}^\varepsilon - U_N^\varepsilon\|_{H^1(D)}^2) = 0.$$

Therefore, $\{U_N^\varepsilon\}$ is a Cauchy sequence in $L^2(\Omega, H^1(D))$. Since $L^2(\Omega, H^1(D))$ is a Banach space, then there exists a function $U^\varepsilon \in L^2(\Omega, H^1(D))$ such that

$$\lim_{N \rightarrow \infty} U_N^\varepsilon = U^\varepsilon \quad \text{in } L^2(\Omega, H^1(D)).$$

To show (ii), we first notice that by the definitions of u_n and U_N^ε , it is easy to check that U_N^ε satisfies

$$\begin{aligned} (3.13) \quad &\int_{\Omega} \left((\nabla U_N^\varepsilon, \nabla v)_D - k^2 (\alpha^2 U_N^\varepsilon, v)_D + \mathbf{i}k \langle \alpha U_N^\varepsilon, v \rangle_{\partial D} \right) dP \\ &= \int_{\Omega} (f, v)_D dP - k^2 \varepsilon^N \int_{\Omega} (\eta(2 + \varepsilon\eta)u_{N-1} + \eta^2 u_{N-2}, v)_D dP \\ &\quad + \mathbf{i}k \varepsilon^N \int_{\Omega} \langle \eta u_{N-1}, v \rangle_{\partial D} dP \end{aligned}$$

for all $v \in L^2(\Omega, H^1(D))$, where $\alpha = 1 + \varepsilon\eta$. In other words, U_N^ε solves the following Helmholtz problem:

$$\begin{aligned} -\Delta U_N^\varepsilon - k^2 \alpha^2 U_N^\varepsilon &= f - k^2 \varepsilon^N (\eta(2 + \varepsilon\eta)u_{N-1} + \eta^2 u_{N-2}) && \text{in } D, \\ \partial_\nu U_N^\varepsilon + \mathbf{i}k \alpha U_N^\varepsilon &= -\mathbf{i}k \varepsilon^N \eta u_{N-1} && \text{on } \partial D. \end{aligned}$$

By (3.6) and the Schwarz inequality we have

$$\begin{aligned} &k^2 \varepsilon^N \left| \int_{\Omega} (\eta(2 + \varepsilon\eta)u_{N-1} + \eta^2 u_{N-2}, v)_D dP \right| \\ &\leq 3k^2 \varepsilon^N \left((\mathbb{E}(\|u_{N-1}\|_{L^2(D)}^2))^{1/2} + (\mathbb{E}(\|u_{N-2}\|_{L^2(D)}^2))^{1/2} \right) (\mathbb{E}(\|v\|_{L^2(D)}^2))^{1/2} \\ &\leq 6k^2 \varepsilon^N \left(\frac{1}{k} + \frac{1}{k^2} \right) C(N-1, k)^{1/2} (\mathbb{E}(\|f\|_{L^2(D)}^2))^{1/2} (\mathbb{E}(\|v\|_{L^2(D)}^2))^{1/2} \\ &\leq 3\varepsilon(k+1)C_0^{1/2} \sigma^{N-1} (\mathbb{E}(\|f\|_{L^2(D)}^2))^{1/2} (\mathbb{E}(\|v\|_{L^2(D)}^2))^{1/2} \\ &\rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad \text{provided that } \sigma < 1. \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 & k\varepsilon^N \left| \int_{\Omega} \langle \eta u_{N-1}, v \rangle_{\partial D} dP \right| \\
 & \leq k\varepsilon^N (\mathbb{E}(\|u_{N-1}\|_{L^2(\partial D)}^2))^{\frac{1}{2}} (\mathbb{E}(\|v\|_{L^2(\partial D)}^2))^{\frac{1}{2}} \\
 & \leq k\varepsilon^N \left(\frac{1}{k} + \frac{1}{k^2} \right) C(N-1, k) (\mathbb{E}(\|f\|_{L^2(D)}^2))^{\frac{1}{2}} (\mathbb{E}(\|v\|_{L^2(\partial D)}^2))^{\frac{1}{2}} \\
 & \leq \frac{\varepsilon}{2} \left(1 + \frac{1}{k} \right) C_0^{\frac{1}{2}} \sigma^{N-1} (\mathbb{E}(\|f\|_{L^2(D)}^2))^{\frac{1}{2}} (\mathbb{E}(\|v\|_{L^2(\partial D)}^2))^{\frac{1}{2}} \\
 & \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad \text{provided that } \sigma < 1.
 \end{aligned}$$

Setting $N \rightarrow \infty$ in (3.13) immediately yields

$$\begin{aligned}
 (3.14) \quad & \int_{\Omega} \left((\nabla U^\varepsilon, \nabla v)_D - k^2 (\alpha^2 U^\varepsilon, v)_D + \mathbf{i}k \langle \alpha U^\varepsilon, v \rangle_{\partial D} \right) dP \\
 & = \int_{\Omega} (f, v)_D dP \quad \forall v \in L^2(\Omega, H^1(D)).
 \end{aligned}$$

Thus, U^ε is a solution to problem (1.1)–(1.2). By the uniqueness of the solution, we conclude that $U^\varepsilon = u^\varepsilon$. Therefore, (3.1) holds in $L^2(\Omega, H^1(D))$. The proof is complete. \blacksquare

The above proof also implies an upper bound for the error $u^\varepsilon - U_N^\varepsilon$, as stated in the next theorem.

Theorem 3.3. *Let U_N^ε be the same as above, and let u^ε denote the solution to problem (1.1)–(1.2) and $\sigma := 4\varepsilon C_0^{\frac{1}{2}}(1+k)$. Then there holds for $\varepsilon(2\varepsilon+1) < \gamma_0$*

$$(3.15) \quad \mathbb{E}(\|u^\varepsilon - U_N^\varepsilon\|_{H^j(D)}^2) \leq \frac{9C_0\sigma^{2N}}{32(1+k)^2} \left(k^j + \frac{1}{k} \right)^4 \mathbb{E}(\|f\|_{L^2(D)}^2), \quad j = 0, 1,$$

provided that $\sigma < 1$, where C is a positive constant independent of k and ε .

Proof. Let $E_N^\varepsilon := u^\varepsilon - U_N^\varepsilon$; subtracting (3.13) from (3.14), we get

$$\begin{aligned}
 (3.16) \quad & \int_{\Omega} \left((\nabla E_N^\varepsilon, \nabla v)_D - k^2 (\alpha^2 E_N^\varepsilon, v)_D + \mathbf{i}k \langle \alpha E_N^\varepsilon, v \rangle_{\partial D} \right) dP \\
 & = k^2 \varepsilon^N \int_{\Omega} (\eta(2 + \varepsilon\eta)u_{N-1} + \eta^2 u_{N-2}, v)_D dP \\
 & \quad - \mathbf{i}k \varepsilon^N \int_{\Omega} \langle \eta u_{N-1}, v \rangle_{\partial D} dP \quad \forall v \in L^2(\Omega, H^1(D)).
 \end{aligned}$$

In other words, E_N^ε solves the following Helmholtz problem:

$$\begin{aligned}
 -\Delta E_N^\varepsilon - k^2 \alpha^2 E_N^\varepsilon &= k^2 \varepsilon^N (\eta(2 + \varepsilon\eta)u_{N-1} + \eta^2 u_{N-2}) & \text{in } D, \\
 \partial_\nu E_N^\varepsilon + \mathbf{i}k \alpha E_N^\varepsilon &= -\mathbf{i}k \varepsilon^N \eta u_{N-1} & \text{on } \partial D.
 \end{aligned}$$

By Theorem 2.4 and (3.6) we obtain for $j = 0, 1$

$$\begin{aligned} \mathbb{E}(\|E_N^\varepsilon\|_{H^j(D)}^2) &\leq 18C_0 \left(k^{j-1} + \frac{1}{k^2}\right)^2 \left[k^4 \varepsilon^{2N} \left(\mathbb{E}(\|u_{N-1}\|_{L^2(D)}^2) + \mathbb{E}(\|u_{N-2}\|_{L^2(D)}^2) \right) \right. \\ &\quad \left. + k^2 \varepsilon^{2N} \mathbb{E}(\|u_{N-1}\|_{L^2(\partial D)}^2) \right] \\ &\leq 18C_0 k^4 \varepsilon^{2N} \left(k^{j-1} + \frac{1}{k^2}\right)^4 C(N-1, k) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ &\leq \frac{18C_0 \sigma^{2N}}{64(1+k)^2} \left(k^j + \frac{1}{k}\right)^4 \mathbb{E}(\|f\|_{L^2(D)}^2). \end{aligned}$$

The proof is complete. ■

4. Monte Carlo discontinuous Galerkin approximations of the mode functions $\{u_n\}$.

In the previous section, we presented a multimodes representation of the solution u^ε and a convergence rate estimate for its finite approximations. These results will serve as the theoretical foundation for our overall numerical methodology for approximating the solution u^ε of problem (1.1)–(1.2). To compute $\mathbb{E}(u^\varepsilon)$ following this approach, we need to compute the expectations $\{\mathbb{E}(u_n)\}$ of the first N mode functions $\{u_n\}_{n=0}^{N-1}$. This requires the construction of an accurate and robust numerical (discretization) method to compute the expectations of the solutions to the “nearly” deterministic Helmholtz problems (3.3), (3.5) and (3.4), (3.5) satisfied by the mode functions $\{u_n\}$. The construction of such a numerical method is precisely our focus in this section. We note that due to the multiplicative structure of the right-hand side of (3.4), $\mathbb{E}(u_n)$ cannot be computed directly for $n \geq 1$. On the other hand, $\mathbb{E}(u_0)$ can be computed directly because it satisfies the deterministic Helmholtz equation with the source term $\mathbb{E}(f)$ and homogeneous boundary condition.

The goal of this section is to develop some *Monte Carlo interior penalty discontinuous Galerkin* (MCIP-DG) methods for the above-mentioned Helmholtz problems. Our MCIP-DG methods are the direct generalizations of the deterministic IP-DG methods proposed in [7, 8] for the related deterministic Helmholtz problems. It should be noted that although various numerical methods (such as finite difference, finite element, and spectral methods) can be used for the job, the IP-DG methods to be presented below are the only general purpose discretization methods which are unconditionally stable (i.e., stable without mesh constraint) and optimally convergent. This is indeed the primary reason why we choose the IP-DG methods as our spatial discretization methods.

4.1. DG notations. Let \mathcal{T}_h be a quasi-uniform partition of D such that $\bar{D} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$. Let h_K denote the diameter of $K \in \mathcal{T}_h$ and $h := \max\{h_K; K \in \mathcal{T}_h\}$. $H^s(\mathcal{T}_h)$ denotes the standard broken Sobolev space and V_r^h denotes the DG finite element space which are defined as

$$H^s(\mathcal{T}_h) := \prod_{K \in \mathcal{T}_h} H^s(K), \quad V_r^h := \prod_{K \in \mathcal{T}_h} P_r(K),$$

where $P_r(K)$ is the set of all polynomials whose degrees do not exceed a given positive integer r . Let \mathcal{E}^I denote the set of all interior faces/edges of \mathcal{T}_h , let \mathcal{E}^B denote the set of all boundary faces/edges of \mathcal{T}_h , and let $\mathcal{E} := \mathcal{E}^I \cup \mathcal{E}^B$. The L^2 -inner product for piecewise defined functions

over the mesh \mathcal{T}_h is naturally defined by

$$(v, w)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \int_K vw \, dx,$$

and for any set $\mathcal{S}_h \subset \mathcal{E}$, the L^2 -inner product over \mathcal{S}_h is defined by

$$\langle v, w \rangle_{\mathcal{S}_h} := \sum_{e \in \mathcal{S}_h} \int_e vw \, dS.$$

Let $K, K' \in \mathcal{T}_h$ and $e = \partial K \cap \partial K'$ and assume the global labeling number of K is smaller than that of K' . We choose $n_e := n_K|_e = -n_{K'}|_e$ as the unit normal on e outward to K and define the following standard jump and average notation across the face/edge e :

$$\begin{aligned} [v] &:= v|_K - v|_{K'} & \text{on } e \in \mathcal{E}^I, & & [v] &:= v & \text{on } e \in \mathcal{E}^B, \\ \{v\} &:= \frac{1}{2}(v|_K + v|_{K'}) & \text{on } e \in \mathcal{E}^I, & & \{v\} &:= v & \text{on } e \in \mathcal{E}^B \end{aligned}$$

for $v \in V_r^h$. We also define the following seminorms on $H^s(\mathcal{T}_h)$:

$$\begin{aligned} |v|_{1,h,D} &:= \|\nabla v\|_{L^2(\mathcal{T}_h)}, \\ \|v\|_{1,h,D} &:= \left(|v|_{1,h,D}^2 + \sum_{e \in \mathcal{E}_h^I} \left(\frac{\gamma_{0,e} r}{h_e} \|[v]\|_{L^2(e)}^2 + \sum_{\ell=1}^{d-1} \frac{\beta_{1,e} r}{h_e} \|[\partial_{\tau_e^\ell} v]\|_{L^2(e)}^2 \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^r \sum_{e \in \mathcal{E}_h^I} \gamma_{j,e} \left(\frac{h_e}{r}\right)^{2j-1} \|[\partial_{n_e}^j v]\|_{L^2(e)}^2 \right) \right)^{\frac{1}{2}}, \\ |||v|||_{1,h,D} &:= \left(\|v\|_{1,h,D}^2 + \sum_{e \in \mathcal{E}_h^I} \frac{h_e}{\gamma_{0,e} r} \|\{\partial_{n_e} v\}\|_{L^2(e)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Here $\{\beta_{1,e}\}$ and $\{\gamma_{j,e}\}$ are called penalty parameters, which are underdetermined positive constants that will be specified in the next section.

4.2. IP-DG method for deterministic Helmholtz problem. In this subsection we consider following deterministic Helmholtz problem and its IP-DG approximations proposed in [7, 8]:

$$(4.1) \quad -\Delta \Phi_0 - k^2 \Phi_0 = F_0 \quad \text{in } D,$$

$$(4.2) \quad \partial_\nu \Phi_0 + \mathbf{i}k \Phi_0 = G_0 \quad \text{on } \partial D.$$

We note that $\Phi_0 = \mathbb{E}(u_0)$ satisfies the above equations with $F_0 = \mathbb{E}(f)$ and $G_0 = 0$. As an interesting byproduct, all the results to be presented in this subsection apply to $\mathbb{E}(u_0)$.

The IP-DG weak formulation for (4.1)–(4.2) is defined (cf. [7, 8]) by seeking $\Phi_0 \in H^1(D) \cap H_{\text{loc}}^{r+1}(D)$ such that

$$(4.3) \quad a_h(\Phi_0, \psi) = (F_0, \psi)_D + (G_0, \psi)_{\partial D} \quad \forall \psi \in H^1(D) \cap H^{r+1}(\mathcal{T}_h),$$

where

$$\begin{aligned}
 (4.4) \quad a_h(\phi, \psi) &:= b_h(\phi, \psi) - k^2(\phi, \psi)_{\mathcal{T}_h} + \mathbf{i}k\langle \phi, \psi \rangle_{\mathcal{E}_h^B} + \mathbf{i} \left(L_1(\phi, \psi) + \sum_{j=0}^r J_j(\phi, \psi) \right), \\
 b_h(\phi, \psi) &:= (\nabla \phi, \nabla \psi)_{\mathcal{T}_h} - \left(\langle \{\partial_n \phi\}, [\psi] \rangle_{\mathcal{E}_h^I} + \langle [\phi], \{\partial_n \psi\} \rangle_{\mathcal{E}_h^I} \right), \\
 L_1(\phi, \psi) &:= \sum_{e \in \mathcal{E}_h^I} \sum_{\ell=1}^{d-1} \beta_{1,e} h_e^{-1} \langle [\partial_{\tau^\ell} \phi], [\partial_{\tau^\ell} \psi] \rangle_e, \\
 J_j(\phi, \psi) &:= \sum_{e \in \mathcal{E}_h^I} \gamma_{j,e} h_e^{2j-1} \langle [\partial_n^j \phi], [\partial_n^j \psi] \rangle_e, \quad j = 0, 1, \dots, r.
 \end{aligned}$$

$\{\beta_{1,e}\}$ and $\{\gamma_{j,e}\}$ are piecewise constant nonnegative functions defined on \mathcal{E}_h^I . $\{\tau^\ell\}_{\ell=1}^{d-1}$ denotes an orthonormal basis of the edge and ∂_{τ^ℓ} denotes the tangential derivative in the direction of τ^ℓ .

Remark 4.1. The L_1 and $\{J_j\}$ terms are called interior penalty terms, and $\{\beta_{1,e}\}$ and $\{\gamma_{j,e}\}$ are called penalty parameters. The two distinct features of the DG sesquilinear form $a_h(\cdot, \cdot)$ are (i) it penalizes not only the jumps of the function values but also the jumps of the tangential derivatives as well the jumps of all normal derivatives up to r th order; (ii) the penalty parameters are pure imaginary numbers with nonnegative imaginary parts.

Following [7, 8] and based on the DG weak formulation (4.3), our IP-DG method for problem (4.1)–(4.2) is defined by seeking $\Phi_0^h \in V_r^h$ such that

$$(4.5) \quad a_h(\Phi_0^h, \psi^h) = (F_0, \psi^h)_D + \langle G_0, \psi^h \rangle_{\partial D} \quad \forall \psi^h \in V_r^h.$$

For the above IP-DG method, it was proved in [7, 8] that the method is unconditionally stable and its solutions satisfy some wave-number-explicit stability estimates. Its solutions also satisfy optimal order (in h) error estimates, which are described below.

Theorem 4.1. *Let $\Phi_0^h \in V_r^h$ be a solution to scheme (4.5); then the following hold:*

(i) *For all $h, k > 0$, there exists a positive constant \hat{C}_0 independent of ε and h such that*

$$(4.6) \quad \|\Phi_0^h\|_{L^2(D)} + \frac{1}{k} \|\Phi_0^h\|_{1,h,D} + \|\Phi_0^h\|_{L^2(\partial D)} \leq \hat{C}_0 C_s \hat{M}(F_0, G_0),$$

where

$$\begin{aligned}
 (4.7) \quad C_s &:= \frac{d-2}{k} + \frac{1}{k^2} + \frac{1}{k^2} \max_{e \in \mathcal{E}_h^I} \left(\frac{r k^2 h_e^2 + r^5}{\gamma_{0,e} h_e^2} + \frac{r}{h_e} \max_{0 \leq j \leq r-1} \sqrt{\frac{\gamma_{j,e}}{\gamma_{j+1,e}}} \right. \\
 &\quad \left. + \frac{r^2}{h_e} + \frac{r^3}{h_e^2} \sqrt{\frac{\beta_{1,e}}{\gamma_{1,e}}} \right),
 \end{aligned}$$

$$(4.8) \quad \hat{M}(F_0, G_0) := \|F_0\|_{L^2(D)} + \|G_0\|_{L^2(\partial D)}.$$

(ii) *If $k^3 h^2 r^{-2} = O(1)$, then there exists a positive constant \hat{C}_0 independent of k and h such that*

$$(4.9) \quad \|\Phi_0^h\|_{L^2(D)} + \|\Phi_0^h\|_{L^2(\partial D)} + \frac{1}{k} \|\Phi_0^h\|_{1,h,D} \leq \hat{C}_0 \left(\frac{1}{k} + \frac{1}{k^2} \right) \hat{M}(F_0, G_0).$$

An immediate consequence of (4.6) is the following unconditional solvability and uniqueness result.

Corollary 4.2. *There exists a unique solution to scheme (4.5) for all $k, h > 0$.*

Theorem 4.3. *Let $\Phi_0^h \in V^h$ solve (4.5), let $\Phi_0 \in H^s(\Omega)$ be the solution of (4.1)–(4.2), and let $\mu = \min\{r+1, s\}$. Suppose $\gamma_{j,e}, \beta_{1,e} > 0$. Let $\gamma_j = \max_{e \in \mathcal{E}^I} \gamma_{j,e}$ and $\lambda = 1 + \frac{1}{\gamma_0}$.*

(i) *For all $h, k > 0$, there exists a positive constant \tilde{C}_0 independent of ε and h such that*

$$(4.10) \quad \|\Phi_0 - \Phi_0^h\|_{1,h,D} \leq \tilde{C}_0 \left(C_r + \frac{k^3 h}{r} C_s \hat{C}_r \right) \frac{h^{\mu-1}}{r^{s-1}} \|\Phi_0\|_{H^s(D)},$$

$$(4.11) \quad \|\Phi_0 - \Phi_0^h\|_{L^2(D)} + \|\Phi_0 - \Phi_0^h\|_{L^2(\partial D)} \leq \tilde{C}_0 \hat{C}_r \left(1 + k^2 C_s \right) \frac{h^\mu}{r^s} \|\Phi_0\|_{H^s(D)},$$

where

$$C_r := \lambda \left(1 + \frac{r}{\gamma_0} + \sum_{j=1}^r r^{2j-1} \gamma_j + \frac{kh}{\lambda r} \right)^{\frac{1}{2}},$$

$$\hat{C}_r := \left(1 + \frac{r}{\gamma_0} + r \gamma_1 + \sum_{j=2}^r r^{2j-2} \gamma_j + \frac{kh}{\lambda r} \right)^{\frac{1}{2}} C_r.$$

(ii) *If $k^3 h^2 r^{-2} = O(1)$, then there exists a positive constant \tilde{C}_0 independent of k and h such that*

$$(4.12) \quad \|\Phi_0 - \Phi_0^h\|_{1,h,D} \leq \frac{\tilde{C}_0 (r + k^2 h) h^{\mu-1}}{r^s} \|\Phi_0\|_{H^s(D)},$$

$$(4.13) \quad \|\Phi_0 - \Phi_0^h\|_{L^2(D)} + \|\Phi_0 - \Phi_0^h\|_{L^2(\partial D)} \leq \frac{\tilde{C}_0 k h^\mu}{r^s} \|\Phi_0\|_{H^s(D)}.$$

Remark 4.2. It was proved in [4] (also by Theorem 2.4 with $\varepsilon = 0$) that

$$\|\Phi_0\|_{H^s(D)} \leq \tilde{C}_0 \left(k^{s-1} + \frac{1}{k} \right) \hat{M}(F_0, G_0), \quad s = 0, 1, 2.$$

It is expected that the following higher order norm estimates also hold (cf. [7] for an explanation):

$$(4.14) \quad \|\Phi_0\|_{H^s(D)} \leq \tilde{C}_0 \left(k^{s-1} + \frac{1}{k} \right) \left(\|F_0\|_{H^{s-2}(D)} + \|G_0\|_{H^{s-\frac{5}{2}}(\partial D)} \right), \quad s \geq 3,$$

provided that F_0, G_0 , and D are sufficiently smooth. In such a case, $\|\Phi_0\|_{H^s(D)}$ in (4.10)–(4.13) can be replaced by the above bound so that explicit constants can be obtained in these estimates.

4.3. MCIP-DG method for approximating $\mathbb{E}(u_n)$ for $n \geq 0$. We recall that each mode function u_n satisfies the following Helmholtz problem:

$$(4.15) \quad -\Delta u_n - k^2 u_n = S_n \quad \text{in } D,$$

$$(4.16) \quad \partial_\nu u_n + \mathbf{i}k u_n = Q_n \quad \text{on } \partial D,$$

where

$$u_{-1} := 0, S_0 := f, Q_0 := 0, S_n := 2k^2\eta u_{n-1} + k^2\eta^2 u_{n-2}, Q_n := -ik\eta u_{n-1}, n \geq 1.$$

Clearly, $S_n(x, \cdot)$ and Q_n are random processes for a.e. $x \in D$, $S_n \in L^2(\Omega, L^2(D))$ and $Q_n \in L^2(\Omega, L^2(\partial D))$. We remark again that due to its multiplicative structure, $\mathbb{E}(S_n)$ and $\mathbb{E}(Q_n)$ can not be computed directly for $n \geq 1$. Otherwise, (4.15) and (4.16) would be easily converted into deterministic equations for $\mathbb{E}(u_n)$, as we did early for $\mathbb{E}(u_0)$. In other words, (4.15)–(4.16) is a genuine random PDE problem. On the other hand, since all the coefficients of the equations are constants, then the problem is nearly deterministic. Such a remarkable property will be fully exploited in our overall numerical methodology, which will be described in the next section.

Several numerical methodologies are well known in the literature for discretizing random PDEs. Monte Carlo Galerkin, stochastic Galerkin (or polynomial chaos), and stochastic collocation are three such methods (cf. [2, 1] and the references therein). Due to the nearly deterministic structure of (4.15)–(4.16), we propose to discretize it using the Monte Carlo IP-DG approach, which combines the classical Monte Carlo method for the stochastic variable and the IP-DG method; we present this in the proceeding subsection for the spatial variable.

Following the standard formulation of the Monte Carlo method (cf. [2]), let M be a (large) positive integer which will be used to denote the number of realizations, and let V_r^h be the DG space defined in section 4.1. For each $j = 1, 2, \dots, M$, we sample i.i.d. realizations of the source term $f(\omega_j, \cdot)$ and random medium coefficient $\eta(\omega_j, \cdot)$, and we recursively find corresponding approximation $u_n^h(\omega_j, \cdot) \in V_r^h$ such that

$$(4.17) \quad a_n(u_n^h(\omega_j, \cdot), \psi^h) = (S_n^h(\omega_j, \cdot), \psi^h)_D + \langle Q_n^h(\omega_j, \cdot), \psi^h \rangle_{\partial D} \quad \forall \psi^h \in V_r^h$$

for $n = 0, 1, 2, \dots, N - 1$, where

$$(4.18) \quad S_0^h(\omega_j, \cdot) := f(\omega_j, \cdot), \quad Q_0^h := 0,$$

$$(4.19) \quad u_{-1}^h(\omega_j, \cdot) := 0,$$

$$(4.20) \quad S_n^h(\omega_j, \cdot) := 2k^2\eta u_{n-1}^h(\omega_j, \cdot) + k^2\eta^2 u_{n-2}^h(\omega_j, \cdot), \quad n = 1, 2, \dots, N - 1,$$

$$(4.21) \quad Q_n^h(\omega_j, \cdot) := -ik\eta u_{n-1}^h(\omega_j, \cdot), \quad n = 1, 2, \dots, N - 1.$$

We point out that in order for u_n^h to be computable, S_n^h and Q_n^h , not S_n and Q_n , are used on the right-hand side of (4.17). This (small) perturbation on the right-hand side will result in an additional discretization error which must be accounted for later; see section 5.

Next, we approximate $\mathbb{E}(u_n)$ by the following sample average:

$$(4.22) \quad \Phi_n^h := \frac{1}{M} \sum_{j=1}^M u_n^h(\omega_j, \cdot).$$

The following lemma is well known (cf. [2, 14]).

Lemma 4.4. *There hold the following estimates for $n \geq 0$:*

$$(4.23) \quad \mathbb{E}(\|\mathbb{E}(u_n^h) - \Phi_n^h\|_{L^2(D)}^2) \leq \frac{1}{M} \mathbb{E}(\|u_n^h\|_{L^2(D)}^2),$$

$$(4.24) \quad \mathbb{E}(\|\mathbb{E}(u_n^h) - \Phi_n^h\|_{1,h,D}^2) \leq \frac{1}{M} \mathbb{E}(\|u_n^h\|_{1,h,D}^2).$$

To bound $\mathbb{E}(\|u_n^h\|_{1,h,D}^2)$, we once again use the induction argument. To avoid some technicalities, we only provide a proof for the case when the mesh size is in the preasymptotic regime, i.e., $k^3 h^2 r^{-2} = O(1)$.

Lemma 4.5. *Assume $k^3 h^2 r^{-2} = O(1)$. Then there hold for $n \geq 0$*

$$(4.25) \quad \mathbb{E}(\|u_n^h\|_{L^2(D)}^2 + \|u_n^h\|_{L^2(\partial D)}^2) \leq \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 \hat{C}(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2),$$

$$(4.26) \quad \mathbb{E}(\|u_n^h\|_{1,h,D}^2) \leq \left(1 + \frac{1}{k}\right)^2 \hat{C}(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2),$$

where

$$(4.27) \quad \hat{C}(0, k) := \hat{C}_0^2, \quad \hat{C}(n, k) := 4^{2n-1} \hat{C}_0^{2n+2} (1+k)^{2n} \quad \text{for } n \geq 1.$$

Proof. By (4.17) and estimate (4.9) we immediately get

$$\begin{aligned} \mathbb{E}(\|u_0^h\|_{L^2(D)}^2 + \|u_0^h\|_{L^2(\partial D)}^2) &\leq \hat{C}_0^2 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 \mathbb{E}(\|S_0^h\|_{L^2(D)}^2) \leq \hat{C}_0^2 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2), \\ \mathbb{E}(\|u_0^h\|_{1,h,D}^2) &\leq \hat{C}_0^2 \left(1 + \frac{1}{k}\right)^2 \mathbb{E}(\|S_0^h\|_{L^2(D)}^2) \leq \hat{C}_0^2 \left(1 + \frac{1}{k}\right)^2 \mathbb{E}(\|f\|_{L^2(D)}^2), \end{aligned}$$

which verifies (4.25) and (4.26) for $n = 0$. Suppose (4.25) and (4.26) hold for all $n = 0, 1, 2, \dots, \ell - 1$; we now prove that they also hold for $n = \ell$.

Again, by (4.17) with $n = \ell - 1$ and estimate (4.9) we have

$$\begin{aligned} \mathbb{E}(\|u_\ell^h\|_{L^2(D)}^2 + \|u_\ell^h\|_{L^2(\partial D)}^2) &\leq \hat{C}_0^2 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 \mathbb{E}(\|S_\ell^h\|_{L^2(D)}^2 + \|Q_n^h\|_{L^2(\partial D)}^2) \\ &\leq 2\hat{C}_0^2 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 k^4 E\left(4\|u_{\ell-1}^h\|_{L^2(D)}^2 + \|u_{\ell-2}^h\|_{L^2(D)}^2 + \frac{1}{k^2}\|u_{\ell-1}^h\|_{L^2(\partial D)}^2\right) \\ &\leq 2\hat{C}_0^2 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 (1+k)^2 \left(4\hat{C}(\ell-1, k) + \hat{C}(\ell-2, k)\right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ &\leq 8\hat{C}_0^2 \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 (1+k)^2 \hat{C}(\ell-1, k) \left(1 + \frac{\hat{C}(\ell-2, k)}{4\hat{C}(\ell-1, k)}\right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ &\leq \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 \hat{C}(\ell, k) \mathbb{E}(\|f\|_{L^2(D)}^2); \end{aligned}$$

here we have used the fact that

$$8\hat{C}_0^2 (1+k)^2 \hat{C}(\ell-1, k) \left(1 + \frac{\hat{C}(\ell-2, k)}{4\hat{C}(\ell-1, k)}\right) \leq \hat{C}(\ell, k).$$

Similarly, we have

$$\begin{aligned} \mathbb{E}(\|u_\ell^h\|_{1,h,D}^2) &\leq \hat{C}_0^2 \left(1 + \frac{1}{k}\right)^2 \mathbb{E}(\|S_\ell^h\|_{L^2(D)}^2 + \|Q_n^h\|_{L^2(\partial D)}^2) \\ &\leq 2\hat{C}_0^2 \left(1 + \frac{1}{k}\right)^2 k^4 E\left(4\|u_{\ell-1}^h\|_{L^2(D)}^2 + \mathbb{E}(\|u_{\ell-2}^h\|_{L^2(D)}^2) + \frac{1}{k^2}\|u_{\ell-1}^h\|_{L^2(\partial D)}^2\right) \\ &\leq 2\hat{C}_0^2 \left(1 + \frac{1}{k}\right)^2 (1+k)^2 \left(4\hat{C}(\ell-1, k) + \hat{C}(\ell-2, k)\right) \mathbb{E}(\|f\|_{L^2(D)}^2) \\ &\leq \left(1 + \frac{1}{k}\right)^2 \hat{C}(\ell, k) \mathbb{E}(\|f\|_{L^2(D)}^2). \end{aligned}$$

This completes the induction argument and the proof. ■

Combining Lemmas 4.4 and 4.5, we have the following.

Theorem 4.6. *Suppose $k^3 h^2 r^{-2} = O(1)$. Then there hold*

$$(4.28) \quad \mathbb{E}(\|\mathbb{E}(u_n^h) - \Phi_n^h\|_{L^2(D)}^2) \leq \frac{1}{M} \left(\frac{1}{k} + \frac{1}{k^2}\right)^2 \hat{C}(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2),$$

$$(4.29) \quad \mathbb{E}(\|\mathbb{E}(u_n^h) - \Phi_n^h\|_{1,h,D}^2) \leq \frac{1}{M} \left(1 + \frac{1}{k}\right)^2 \hat{C}(n, k) \mathbb{E}(\|f\|_{L^2(D)}^2).$$

Remark 4.3. Estimates (4.28) and (4.29) show that for each fixed $n \geq 0$ the statistical error due to sampling is controlled by the number of realizations of u_n^h . Indeed, it can be easily proved by using Markov’s inequality and the Borel–Cantelli lemma that the statistical error converges to zero as M tends to infinity; see [2, Proposition 4.1] and [14, Theorem 3.2].

5. The overall numerical procedure.

5.1. The numerical algorithm, linear solver, and computational complexity.

We are now ready to introduce our overall numerical procedure for approximating the solution of the original random Helmholtz problem (1.1)–(1.2). Our numerical procedure consists of three main ingredients. First, it is based on the multimodes representation (3.1) and its finite modes approximation (3.12). Second, it uses the classical Monte Carlo method for sampling the probability space and for computing the expectations of the numerical solutions. Finally, at each realization an IP-DG method is employed to solve all the involved deterministic Helmholtz problems. The precise description of this procedure is given by the following algorithm.

Main Algorithm

Inputs: $f, \eta, \varepsilon, k, h, M, N$.

Set $\Psi_N^h(\cdot) = 0$ (initializing).

For $j = 1, 2, \dots, M$

Set $S_0^h(\omega_j, \cdot) = f(\omega_j, \cdot)$.

Set $Q_0^h(\omega_j, \cdot) = 0$.

Set $u_{-1}^h(\omega_j, \cdot) = 0$.

Set $U_N^h(\omega_j, \cdot) = 0$ (initializing).

For $n = 0, 1, \dots, N - 1$

Solve for $u_n^h(\omega_j, \cdot) \in V_r^h$ such that

$$a_h(u_n^h(\omega_j, \cdot), v_h) = (S_n^h(\omega_j, \cdot), v_h)_D + \langle Q_n^h(\omega_j, \cdot), v_h \rangle_{\partial D} \quad \forall v_h \in V_r^h.$$

Set $U_N^h(\omega_j, \cdot) \leftarrow U_N^h(\omega_j, \cdot) + \varepsilon^n u_n^h(\omega_j, \cdot)$.

Set $S_{n+1}^h(\omega_j, \cdot) = 2k^2 \eta(\omega_j, \cdot) u_n^h(\omega_j, \cdot) + k^2 \eta(\omega_j, \cdot)^2 u_{n-1}^h(\omega_j, \cdot)$.

Set $Q_{n+1}^h(\omega_j, \cdot) = -ik \eta(\omega_j, \cdot) u_n^h(\omega_j, \cdot)$.

Endfor

Set $\Psi_N^h(\cdot) \leftarrow \Psi_N^h(\cdot) + \frac{1}{M} U_N^h(\omega_j, \cdot)$.

Endfor

Output $\Psi_N^h(\cdot)$.

We remark that Φ_n^h , defined in (4.22), does not appear in the algorithm. But it is easy to see that

$$(5.1) \quad \Psi_N^h = \Phi_0^h + \varepsilon \Phi_1^h + \varepsilon^2 \Phi_2^h + \cdots + \varepsilon^{N-1} \Phi_{N-1}^h.$$

It is also easy to see that computationally the most expensive steps in the above algorithm are those in the inside loop. In each step of the loop, one is required to solve a large (especially for large k), ill-conditioned, indefinite, and non-Hermitian complex linear system. It is well known that none of iterative methods works well for solving such a linear system (cf. [10]). Moreover, the algorithm requires one to solve a total of MN numbers of such complex linear systems. Such a task is not feasible on most present day computers. However, instead of using such a brute-force approach, we notice that all these MN complex linear systems share the *same* constant coefficient matrix. The systems only differ in their right-hand side vectors! This is an ideal setup for using the LU direct solver. Namely, we only need to perform one LU decomposition of the coefficient matrix and save it. The decomposition can be reused to solve the remaining $MN - 1$ complex linear systems by performing $MN - 1$ sets of forward and backward substitutions. This indeed is the main advantage of the numerical procedure proposed in this paper.

The computational complexity of the above algorithm can be calculated as follows. Let h denote the mesh size of \mathcal{T}_h and $K := \frac{1}{h}$ (assume it is a positive integer). Then the (common) coefficient matrix appearing in the algorithm has the size $O(K^d \times K^d)$, where d denotes the spatial dimension of the domain D . Thus, one LU decomposition requires $O(\frac{3K^{3d}}{2})$ multiplications. All $(MN - 1)$ sets of forward and backward substitutions contribute $O(MNK^{2d})$ multiplications. Since N is a relatively small number in practice, it can be treated as a constant. In particular, to obtain the equal order L^2 -norm error estimates for the IP-DG method and the Monte Carlo simulation (see Theorem 5.5), we need to have $M = K^4$. Then the total cost for implementing the above main algorithm is $O(MNK^{2d}) = O(K^{2d+4})$. As a comparison, we note that if a brute-force Monte Carlo method with $M = K^4$ is used to solve (1.1)–(1.2), it requires $O(\frac{3K^{3d+4}}{2})$ multiplications. Finally, we remark that the outer loop of the main algorithm can be naturally implemented in parallel.

5.2. Convergence analysis. In this subsection, we shall combine the error estimates derived in the previous subsections for various steps in the main algorithm to obtain error estimates for the global error $\mathbb{E}(u^\varepsilon) - \Psi_N^h$. To this end, we notice that $\mathbb{E}(u^\varepsilon) - \Psi_N^h$ can be decomposed as

$$\mathbb{E}(u^\varepsilon) - \Psi_N^h = (\mathbb{E}(u^\varepsilon) - \mathbb{E}(U_N^\varepsilon)) + (\mathbb{E}(U_N^\varepsilon) - \mathbb{E}(U_N^h)) + (\mathbb{E}(U_N^h) - \Psi_N^h).$$

Clearly, the first term on the right-hand side measures the finite modes representation error, the second term measures the spatial discretization error, and the third term represents the statistical error due to the Monte Carlo method.

First, by (3.15) the finite modes representation error can be bounded as follows:

$$(5.2) \quad \mathbb{E}(\|u^\varepsilon - U_N^\varepsilon\|_{H^j(D)}^2) \leq \frac{C_0 \sigma^{2N}}{4(1+k)^2} \left(k^j + \frac{1}{k}\right)^4 \mathbb{E}(\|f\|_{L^2(D)}^2), \quad j = 0, 1,$$

where $\sigma := 4\varepsilon C_0^{\frac{1}{2}}(1+k)$.

Next, we note that

$$U_N^h - \Psi_N^h = \sum_{n=0}^{N-1} \varepsilon^n (u_n^h - \Phi_n^h).$$

Then by (4.28) we bound the statistical error as follows:

$$\begin{aligned} (5.3) \quad \mathbb{E}(\|E(U_N^h) - \Psi_N^h\|_{L^2(D)}) &\leq \sum_{n=0}^{N-1} \varepsilon^n \mathbb{E}(\|E(u_n^h) - \Phi_n^h\|_{L^2(D)}) \\ &\leq \frac{1}{\sqrt{M}} \left(\frac{1}{k} + \frac{1}{k^2}\right) \|f\|_{L^2(\Omega, L^2(D))} \sum_{n=0}^{N-1} \varepsilon^n \hat{C}(n, k)^{\frac{1}{2}} \\ &\leq \frac{\hat{C}_0}{2\sqrt{M}} \left(\frac{1}{k} + \frac{1}{k^2}\right) \|f\|_{L^2(\Omega, L^2(D))} \sum_{n=0}^{N-1} 4^n \varepsilon^n \hat{C}_0^n (1+k)^n \\ &\leq \frac{\hat{C}_0}{2\sqrt{M}} \left(\frac{1}{k} + \frac{1}{k^2}\right) \|f\|_{L^2(\Omega, L^2(D))} \cdot \frac{1}{1-\hat{\sigma}}, \end{aligned}$$

where $\hat{\sigma} := 4\varepsilon \hat{C}_0(1+k) < 1$.

Similarly, by (4.29) we get

$$\begin{aligned} (5.4) \quad \mathbb{E}(\|E(U_N^h) - \Psi_N^h\|_{1,h,D}) &\leq \sum_{n=0}^{N-1} \varepsilon^n \mathbb{E}(\|E(u_n^h) - \Phi_n^h\|_{1,h,D}) \\ &\leq \frac{\hat{C}_0}{2\sqrt{M}} \left(1 + \frac{1}{k}\right) \|f\|_{L^2(\Omega, L^2(D))} \cdot \frac{1}{1-\hat{\sigma}}. \end{aligned}$$

Finally, to bound the spatial discretization error, we recall that $u_n^h \in V_r^h$ is defined by (cf. (4.17))

$$(5.5) \quad a_h(u_n^h, \psi^h) = (S_n^h, \psi^h)_D + \langle Q_n^h, \psi^h \rangle_{\partial D} \quad \forall \psi^h \in V_r^h, \text{ a.s.}$$

for $n \geq 0$. We also define $\tilde{u}_n^h \in V_r^h$ for $n \geq 0$ by

$$(5.6) \quad a_h(\tilde{u}_n^h, \psi^h) = (S_n, \psi^h)_D + \langle Q_n, \psi^h \rangle_{\partial D} \quad \forall \psi^h \in V_r^h, \text{ a.s.}$$

Notice that the difference between u_n^h and \tilde{u}_n^h is that S_n^h and Q_n^h are used in (5.5), while S_n and Q_n are used in (5.6). Corollary 4.2 guarantees that $\{u_n^h\}$ and $\{\tilde{u}_n^h\}$ are uniquely defined.

It follows from Theorem 4.3(ii) that for $k^3 h^2 r^{-2} = O(1)$ there hold

$$(5.7) \quad \mathbb{E}(\|u_n - \tilde{u}_n^h\|_{1,h,D}) \leq \frac{\tilde{C}_0(r+k^2h)h^{\mu-1}}{r^s} \mathbb{E}(\|u_n\|_{H^s(D)}),$$

$$(5.8) \quad \mathbb{E}(\|u_n - \tilde{u}_n^h\|_{L^2(D)} + \|u_n - \tilde{u}_n^h\|_{L^2(\partial D)}) \leq \frac{\tilde{C}_0 k h^\mu}{r^s} \mathbb{E}(\|u_n\|_{H^s(D)}),$$

where $\mu = \min\{r+1, s\}$.

To bound $\tilde{u}_n^h - u_n^h$, we subtract (5.5) from (5.6) to get

$$a_h(\tilde{u}_n^h - u_n^h, \psi^h) = (S_n - S_n^h, \psi^h)_D + \langle Q_n - Q_n^h, \psi^h \rangle_{\partial D} \quad \forall \psi^h \in V_r^h, \text{ a.s.}$$

Then by Theorem 4.1(ii) we get

$$\begin{aligned} (5.9) \quad \mathbb{E} & \left(k \|\tilde{u}_n^h - u_n^h\|_{L^2(D)} + k \|\tilde{u}_n^h - u_n^h\|_{L^2(\partial D)} + \|\tilde{u}_n^h - u_n^h\|_{1,h,D} \right) \\ & \leq \hat{C}_0 \left(1 + \frac{1}{k} \right) \mathbb{E} \left(\|S_n - S_n^h\|_{L^2(D)} + \|Q_n - Q_n^h\|_{L^2(\partial D)} \right) \\ & \leq 2\tilde{C}_0 k(k+1) E \left(2\|u_{n-1} - u_{n-1}^h\|_{L^2(D)} + \|u_{n-2} - u_{n-2}^h\|_{L^2(D)} \right. \\ & \quad \left. + \frac{1}{k} \|u_{n-1} - u_{n-1}^h\|_{L^2(\partial D)} \right). \end{aligned}$$

It follows from the triangle inequality, (5.7)–(5.9), and the inverse inequality that

$$\begin{aligned} (5.10) \quad \mathbb{E} & \left(\|u_n - u_n^h\|_{L^2(D)} + \|u_n - u_n^h\|_{L^2(\partial D)} \right) \\ & \leq \mathbb{E} \left(\|\tilde{u}_n^h - u_n^h\|_{L^2(D)} + \|\tilde{u}_n^h - u_n^h\|_{L^2(\partial D)} + \|u_n - \tilde{u}_n^h\|_{L^2(D)} \right. \\ & \quad \left. + \|u_n - \tilde{u}_n^h\|_{L^2(\partial D)} \right) \\ & \leq 2\tilde{C}_0(k+1) E \left(2\|u_{n-1} - u_{n-1}^h\|_{L^2(D)} + \|u_{n-2} - u_{n-2}^h\|_{L^2(D)} \right. \\ & \quad \left. + \frac{1}{k} \|u_{n-1} - u_{n-1}^h\|_{L^2(\partial D)} \right) + \frac{\tilde{C}_0 k h^\mu}{r^s} \mathbb{E}(\|u_n\|_{H^s(D)}), \\ (5.11) \quad \mathbb{E} & \left(\|u_n - u_n^h\|_{1,h,D} \right) \leq \mathbb{E} \left(\|\tilde{u}_n^h - u_n^h\|_{1,h,D} + \|u_n - \tilde{u}_n^h\|_{1,h,D} \right) \\ & \leq C h^{-1} \mathbb{E} \left(\|\tilde{u}_n^h - u_n^h\|_{L^2(D)} \right) + \mathbb{E} \left(\|u_n - \tilde{u}_n^h\|_{1,h,D} \right) \\ & \leq C\tilde{C}_0 h^{-1} (k+1) E \left(2\|u_{n-1} - u_{n-1}^h\|_{L^2(D)} + \|u_{n-2} - u_{n-2}^h\|_{L^2(D)} \right. \\ & \quad \left. + \frac{1}{k} \|u_{n-1} - u_{n-1}^h\|_{L^2(\partial D)} \right) + \frac{\tilde{C}_0 (r + k^2 h) h^{\mu-1}}{r^s} \mathbb{E}(\|u_n\|_{H^s(D)}) \end{aligned}$$

for $n \geq 1$.

So we obtain two recursive relations between the spatial errors of consecutive mode functions. Then we want to derive some estimates for the spatial error of each mode function. To this end, we first notice that

$$(5.12) \quad \mathbb{E}(\|u_{-1} - u_{-1}^h\|_{L^2(D)}) = \mathbb{E}(\|u_{-1} - u_{-1}^h\|_{1,h,D}) = 0,$$

$$(5.13) \quad \mathbb{E}(\|u_0 - u_0^h\|_{L^2(D)} + \|u_0 - u_0^h\|_{L^2(\partial D)}) \leq \frac{\tilde{C}_0 k h^\mu}{r^s} \mathbb{E}(\|u_0\|_{H^s(D)}),$$

$$(5.14) \quad \mathbb{E}(\|u_0 - u_0^h\|_{1,h,D}) \leq \frac{\tilde{C}_0 (r + k^2 h) h^{\mu-1}}{r^s} \mathbb{E}(\|u_0\|_{H^s(D)}).$$

The last two inequalities hold because $S_0 = S_0^h$, $Q_0 = 0$, and $\tilde{u}_0^h = u_0^h$. The above estimates for the spatial errors of the approximations of the two starting mode functions allow us to

derive the desired estimates from (5.10) and (5.11) for all mode functions, which will be based on the following simple lemma.

Lemma 5.1. *Let $\gamma, \beta > 0$ be two real numbers, and let $\{c_n\}_{n \geq 0}$ and $\{\alpha_n\}_{n \geq 0}$ be two sequences of nonnegative numbers such that*

$$(5.15) \quad c_0 \leq \gamma \alpha_0, \quad c_n \leq \beta c_{n-1} + \gamma \alpha_n \quad \text{for } n \geq 1.$$

Then there holds

$$(5.16) \quad c_n \leq \gamma \sum_{j=0}^n \beta^{n-j} \alpha_j \quad \text{for } n \geq 1.$$

We omit the proof because it is trivial.

Lemma 5.2. *Suppose $\sigma, \hat{\sigma} < 1$ and $k^3 h^2 r^{-2} = O(1)$. Then there hold*

$$(5.17) \quad \mathbb{E}(\|u_n - u_n^h\|_{L^2(D)} + \|u_n - u_n^h\|_{L^2(\partial D)}) \leq \frac{\tilde{C}_0 k h^\mu}{r^s} \sum_{j=0}^n (2k + 3)^{n-j} \mathbb{E}(\|u_j\|_{H^s(D)}),$$

$$(5.18) \quad \mathbb{E}(\|u_n - u_n^h\|_{1,h,D}) \leq \frac{C \tilde{C}_0^2 k (1+k) h^{\mu-1}}{r^s} \sum_{j=0}^n (2k + 3)^{n-j} \mathbb{E}(\|u_j\|_{H^s(D)}).$$

Proof. Define

$$\begin{aligned} u_{-2} &= u_{-1} = u_{-2}^h = u_{-1}^h = 0, \\ c_n &:= \mathbb{E}(\|u_n - u_n^h\|_{L^2(D)} + \|u_{n-1} - u_{n-1}^h\|_{L^2(D)}) \\ &\quad + \mathbb{E}(\|u_n - u_n^h\|_{L^2(\partial D)} + \|u_{n-1} - u_{n-1}^h\|_{L^2(\partial D)}), \\ \beta &:= 2k + 3, \quad \gamma := \frac{\tilde{C}_0 k h^\mu}{r^s}, \quad \alpha_n := \mathbb{E}(\|u_n\|_{H^s(D)}). \end{aligned}$$

Then by (5.10) we obtain (5.15). Hence (5.17) holds. Inequality (5.18) follows from combining (5.11) and (5.17). The proof is complete. ■

Finally, by the definitions of U_N^ε and U_N^h and from (5.17) and (5.18), we immediately have the following theorem.

Theorem 5.3. *Assume that $u_n \in L^2(\Omega, H^s(D))$ for $n \geq 0$. Then the spatial error $U_N^\varepsilon - U_N^h$ satisfies the following estimates:*

$$(5.19) \quad \mathbb{E}(\|U_N^\varepsilon - U_N^h\|_{L^2(D)}) \leq \frac{\tilde{C}_0 k h^\mu}{r^s} \sum_{n=0}^{N-1} \sum_{j=0}^n \varepsilon^n (2k + 3)^{n-j} \mathbb{E}(\|u_j\|_{H^s(D)}),$$

$$(5.20) \quad \begin{aligned} \mathbb{E}(\|U_N^\varepsilon - U_N^h\|_{1,h,D}) &\leq \frac{C \tilde{C}_0^2 k (1+k) h^{\mu-1}}{r^s} \sum_{n=0}^{N-1} \sum_{j=0}^n \varepsilon^n (2k + 3)^{n-j} \mathbb{E}(\|u_j\|_{H^s(D)}). \end{aligned}$$

To simplify the above spatial error estimates, we need to bound $\mathbb{E}(\|u_n\|_{H^s(D)})$ in terms of higher order norms of f . This is achievable using (4.14) and the three-term recursive relation for $\{u_n\}$. Below we only consider the case when $s = 2$ and leave the general case for the interested reader to explore.

When $s = 2$, the required estimates have been obtained in (3.9). Consequently, we have the following.

Theorem 5.4. *Let $s = 2$. Assume that $u_n \in L^2(\Omega, H^2(D))$ for $n \geq 0$ and $2\sqrt{C_0}(2k+3)\varepsilon < 1$. Then there hold*

$$(5.21) \quad \mathbb{E}(\|U_N^\varepsilon - U_N^h\|_{L^2(D)}) \leq C_3(N, k, \varepsilon) h^2 \|f\|_{L^2(\Omega, L^2(D))},$$

$$(5.22) \quad \mathbb{E}(\|U_N^\varepsilon - U_N^h\|_{1,h,D}) \leq C_4(N, k, \varepsilon) h \|f\|_{L^2(\Omega, L^2(D))},$$

where

$$(5.23) \quad C_3(N, k, \varepsilon) := \frac{\tilde{C}_0 k}{r^2} \cdot \frac{C_0(k^3 + 1)}{k^2(2\sqrt{C_0} - 1)} \cdot \frac{1 - (2\sqrt{C_0}(2k+3)\varepsilon)^N}{1 - 2\sqrt{C_0}(2k+3)\varepsilon},$$

$$(5.24) \quad C_4(N, k, \varepsilon) := \frac{C\tilde{C}_0^2 k(1+k)}{r^2} \cdot \frac{C_0(k^3 + 1)}{k^2(2\sqrt{C_0} - 1)} \cdot \frac{1 - (2\sqrt{C_0}(2k+3)\varepsilon)^N}{1 - 2\sqrt{C_0}(2k+3)\varepsilon}.$$

Proof. By (3.9) and the definition of $C(j, k)$ we get

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{j=0}^n \varepsilon^n (2k+3)^{n-j} \mathbb{E}(\|u_j\|_{H^s(D)}) \\ & \leq \left(k + \frac{1}{k^2}\right) \|f\|_{L^2(\Omega, L^2(D))} \sum_{n=0}^{N-1} \sum_{j=0}^n \varepsilon^n (2k+3)^{n-j} C(j, k)^{\frac{1}{2}} \\ & = \frac{C_0^{\frac{1}{2}}(k^3 + 1)}{2k^2} \|f\|_{L^2(\Omega, L^2(D))} \sum_{n=0}^{N-1} \sum_{j=0}^n \varepsilon^n 4^j C_0^{\frac{j}{2}} (1+k)^j (2k+3)^{n-j} \\ & \leq \frac{C_0(k^3 + 1)}{k^2(2\sqrt{C_0} - 1)} \cdot \frac{1 - (2\sqrt{C_0}(2k+3)\varepsilon)^N}{1 - 2\sqrt{C_0}(2k+3)\varepsilon} \|f\|_{L^2(\Omega, L^2(D))}. \end{aligned}$$

The above inequality and (5.19) yield (5.21). Similarly, the above inequality and (5.20) give (5.22). The proof is complete. \blacksquare

Combining (5.2)–(5.4), (5.21), (5.22), (4.28), and (4.29) we get the following.

Theorem 5.5. *Under the assumptions that $u_n \in L^2(\Omega, H^2(D))$ for $n \geq 0$, $k^3 h^2 r^{-2} = O(1)$ and $2\sqrt{C_0}(2k+3)\varepsilon < 1$, there hold*

$$(5.25) \quad \mathbb{E}(\|\mathbb{E}(u^\varepsilon) - \Psi_N^h\|_{L^2(D)}) \leq C_1 \varepsilon^N + C_2 h^2 + C_3 M^{-\frac{1}{2}},$$

$$(5.26) \quad \mathbb{E}(\|\mathbb{E}(u^\varepsilon) - \Psi_N^h\|_{H^1(D)}) \leq C_4 \varepsilon^N + C_5 h + C_6 M^{-\frac{1}{2}},$$

where $C_j = C_j(C_0, \hat{C}_0, k, \varepsilon)$ are positive constants for $j = 1, 2, \dots, 6$.

Remark 5.1. In previous discussions, without loss of generality it has been assumed that the domain $D \subset B_1(0)$. For a generic star-shaped domain $D \subset B_R(0)$, a simple rescaling argument implies that the above convergence results hold if $(kR)^3 h^2 r^{-2} = O(1)$ and $2\sqrt{C_0}(2kR+3)\varepsilon < 1$.

6. Numerical experiments. In this section we present a series of numerical experiments in order to accomplish the following:

- compare our MCIP-DG method using the multimodes expansion to a classical MCIP-DG method;
- illustrate examples using our MCIP-DG method in which the perturbation parameter ε satisfies the constraint required by the convergence theory;
- illustrate examples using our MCIP-DG method in which the perturbation parameter constraint is violated;
- illustrate examples using our MCIP-DG method in which we allow η to be large in magnitude.

In all our numerical experiments we use the spatial domain $D = (-0.5, 0.5)^2$. To partition D we use a uniform triangulation \mathcal{T}_h . For a positive integer n , $\mathcal{T}_{1/n}$ denotes the triangulation of D consisting of $2n^2$ congruent isosceles triangles with side lengths $1/n$, $1/n$, and $\sqrt{2}/n$. Figure 1 gives the sample triangulation $\mathcal{T}_{1/10}$.

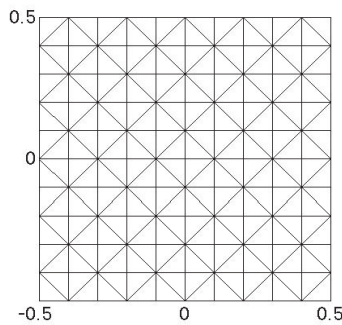


Figure 1. Triangulation $\mathcal{T}_{1/10}$.

For each test we use the DG finite element space V_1^h consisting of piecewise linear functions over the mesh \mathcal{T}_h . The penalty parameters used in (4.4) are

$$\gamma_{1,e} = 0.1, \quad \gamma_{0,e} = (k^2 h)^{\frac{2}{3}} \gamma_{1,e}^{\frac{1}{3}}, \quad \beta_{1,e} = 1.$$

These penalty parameters were shown to be favorable when using the IP-DG method (4.3) to approximate the solution of the deterministic Helmholtz equation with large k (cf. [7]).

To implement the random noise η , we note that η only appears in the integration component of our computations. Therefore, we made the choice to implement η only at quadrature points of the triangulation. To simulate the random media, we let η be an independent random number chosen from a uniform distribution on some closed interval at each quadrature point. Figure 2 shows an example of such random media.

6.1. MCIP-DG with multimodes expansion compared to classical MCIP-DG. The goal of this subsection is to verify the accuracy and efficiency of the proposed MCIP-DG with the multimodes expansion. As a benchmark we compare this method to the classical MCIP-DG (i.e., without utilizing the multimodes expansion). Throughout this section $\tilde{\Psi}^h$ is used to

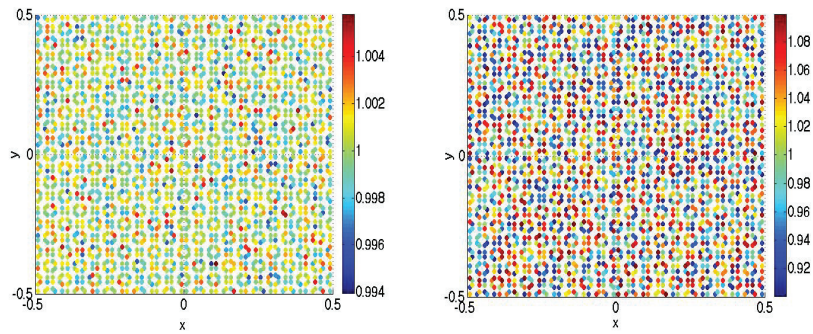


Figure 2. Discrete average media $\frac{1}{M} \sum_{j=1}^M \alpha(\omega_j, \cdot)$ (left) and a sample media $\alpha(\omega, \cdot)$ (right) computed for $h = 1/20$, $\varepsilon = 0.1$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

denote the computed approximation to $\mathbb{E}(u)$ using the classical MCIP-DG. It is well known that the associated error estimate is given by

$$\mathbb{E}(\|\mathbb{E}(u^\varepsilon) - \tilde{\Psi}^h\|_{L^2(D)}) \leq C_1 h^2 + C_2 M^{-\frac{1}{2}}.$$

Below we shall use $\tilde{\Psi}^h$ as a reference solution for comparison with the solution obtained by the multimodes MCIP-DG.

In this subsection we set $f = 1$, $g = 0$, $k = 5$, $1/h = 50$, $M = 1000$, and $\varepsilon = 1/(k + 1)$. The spatial discretization and the number of realizations are the same for both multimodes and classical MCIP-DG methods. Here ε is chosen with the intent of satisfying the constraint set by the convergence theory in the preceding section. η is sampled as described above from a uniform distribution on the interval $[0, 1]$. Ψ_N^h is computed for $N = 1, 2, 3, 4, 5$.

In our first test we compute $\|\Psi_N^h - \tilde{\Psi}^h\|_{L^2(D)}$. The results are displayed in Figure 3. As expected, we find that the difference between Ψ_N^h and $\tilde{\Psi}^h$ is very small. We also observe that only the first couple of modes contribute significantly to the approximation.

To test the efficiency of our MCIP-DG method with multimodes expansion, we compare the CPU time for computing Ψ_N^h and $\tilde{\Psi}^h$. Both methods are implemented on the same computer using MATLAB. MATLAB's built-in LU factorization is called to solve the linear systems. The results of this test are shown in Table 1. As expected, we find that the use of the multimodes expansion improves the CPU time for the computation considerably. In fact, the table shows that this improvement is an order of magnitude. Also, as expected, as the number of modes used is increased, the CPU time increases in a linear fashion.

6.2. More numerical tests. The goal of this subsection is to demonstrate the approximations obtained by our MCIP-DG method with multimodes expansion using different magnitudes of parameter ε and different magnitudes of the random noise η . We only consider the case $0 < \varepsilon < 1$ in order to legitimize the series expansion u^ε . With this in mind, we then increase the magnitude of η to simulate examples with large noise. Similar to the numerical experiments from [7], we choose $f = \sin(k\alpha(\omega, \cdot)r)/r$, where r is the radial distance from the origin, and $g = 0$. In our first set of numerical tests the medium refractive index $\alpha(\omega, \cdot)$ is implemented as described in the beginning of section 6. Since our intention is to observe what happens as we vary ε and N , for this set of tests we fix $k = 50$, $h = 1/100$, and $M = 1000$.

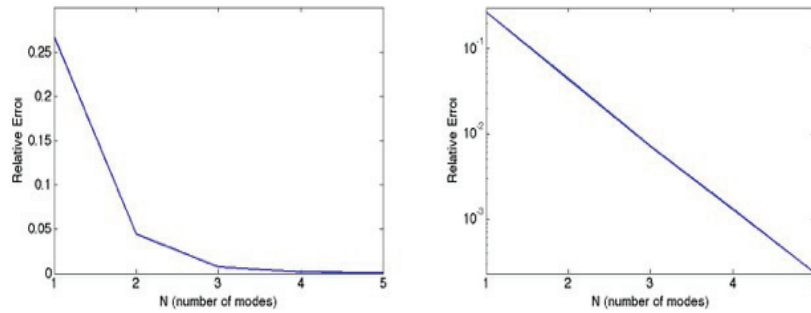


Figure 3. L^2 -norm error between Ψ_N^h computed using MCIP-DG with the multimodes expansion and $\tilde{\Psi}^h$ computed using the classical MCIP-DG. The figure on the left uses a log scale for the L^2 -norm error.

Table 1

CPU times required to compute the MCIP-DG multimodes approximation Ψ_N^h and classical MCIP-DG approximation $\tilde{\Psi}^h$.

Approximation	CPU time (s)
$\tilde{\Psi}^h$	3.4954×10^5
Ψ_1^h	1.0198×10^4
Ψ_2^h	2.0307×10^4
Ψ_3^h	3.0037×10^4
Ψ_4^h	3.9589×10^4
Ψ_5^h	4.9011×10^4

In Figures 4 and 5, we set $\varepsilon = 0.02$ and $|\eta| \leq 1$ with the intent of observing the constraints set in the convergence theory from the preceding section. In Figure 4 we present plots of the magnitude of the computed mean $\text{Re}(\Psi_3^h)$ and a computed sample $\text{Re}(U_3^h)$, respectively, over the whole domain D . Figure 5 gives the plots of a cross section of the computed mean $\text{Re}(\Psi_3^h)$ and a computed sample $\text{Re}(U_3^h)$, respectively, over the line $y = x$. In this first example we observe that the computed sample does not differ greatly from the computed mean because ε is very small.

In Figures 6–11, we fix $|\eta| \leq 1$ and increase ε past the constraint established in the preceding convergence theory. As expected, we see that as ε increases the computed sample differs more from the computed mean. We also observe that as ε increases, the phase of the wave remains relatively intact but the magnitude of the wave becomes more uniform.

In Table 2 the relative error (measured in the L^2 -norm) between the multimodes approximation Ψ_N^h and the classical MCIP-DG approximation $\tilde{\Psi}^h$ is given for $\varepsilon = 0.02, 0.1, 0.5, 0.8$. In this table only three modes (i.e., $N = 3$) are used. Recall that the convergence theory in this case only holds for ε on the order of the first value 0.02. That being said, we observe that the approximations corresponding to $\varepsilon = 0.1$ and $\varepsilon = 0.5$ are relatively close to those obtained using the classical Monte Carlo method. Another observation that can be made from Table 2 is that as ε increases, the relative error increases. This is expected from the convergence theory.

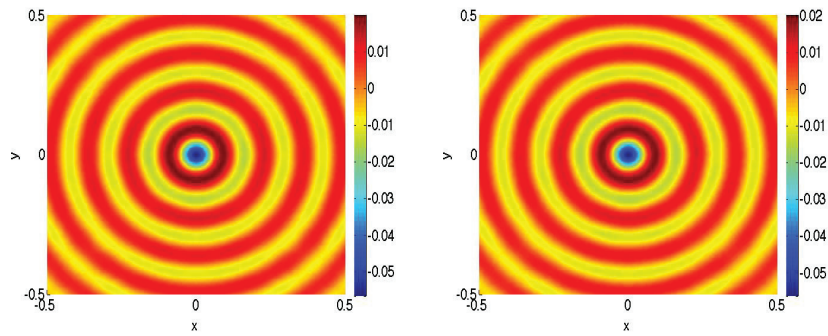


Figure 4. $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.02$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

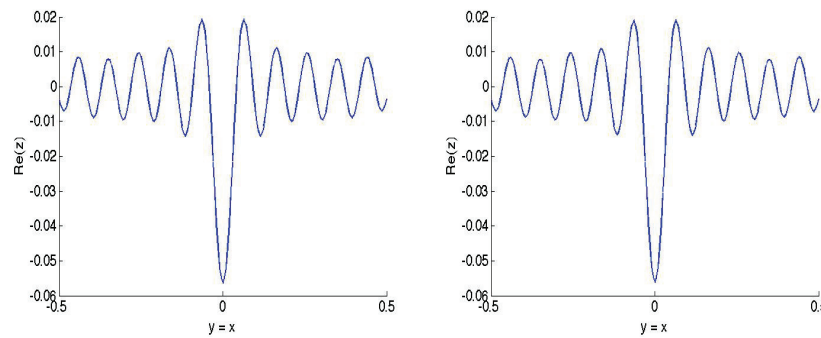


Figure 5. Cross sections of $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.02$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$, over the line $y = x$.

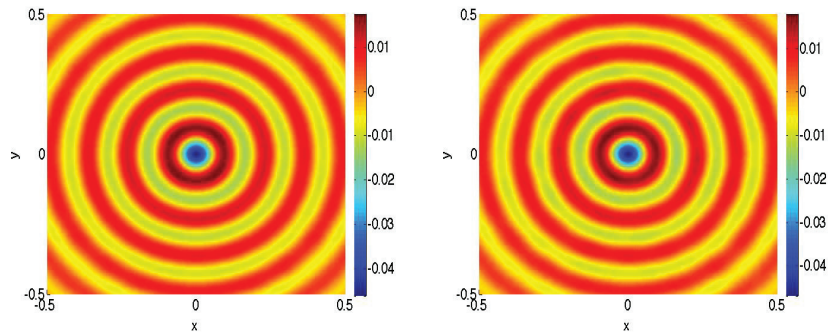


Figure 6. $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.1$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

Recall that the error predicted in the convergence theory can be bounded by a term with the factor ε^N . Thus for ε relatively large, one must use more modes to decrease the error. Keeping this in mind, Table 3 records the relative error (measured in the L^2 -norm) between the multimodes approximation Ψ_N^h and the classical Monte Carlo approximation $\tilde{\Psi}^h$

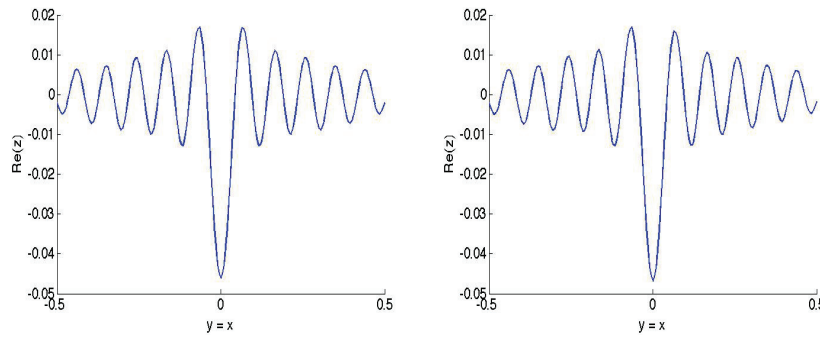


Figure 7. Cross sections of $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.1$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

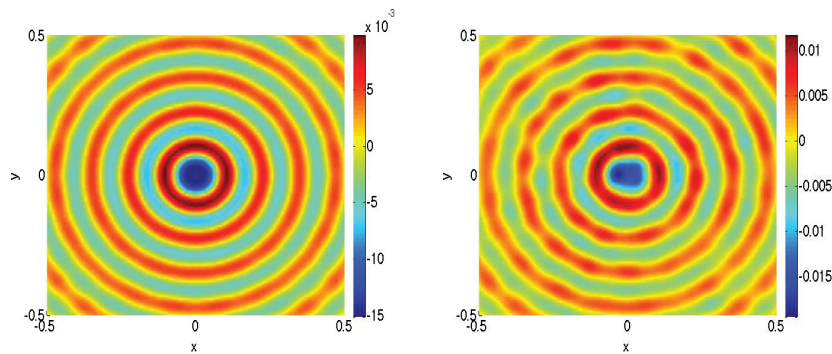


Figure 8. $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.5$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

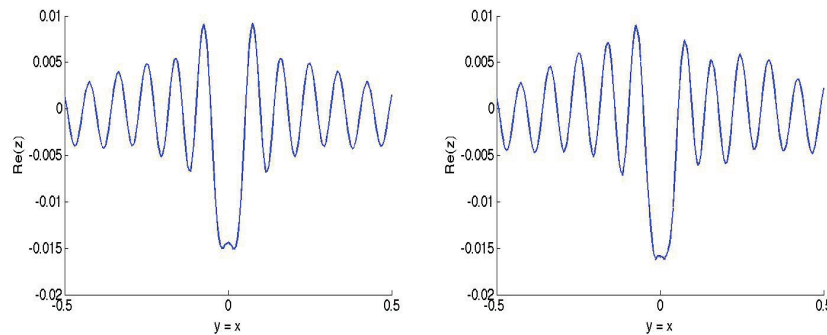


Figure 9. Cross sections of $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.5$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

for $\varepsilon = 0.5, 0.8$ and $N = 4, 5, 6, 7$. At this point, we observe that the relative error decreases as N increases when $\varepsilon = 0.5$. On the other hand, the relative error increases as N increases when $\varepsilon = 0.8$. From Tables 2 and 3 we observe that multimodes expansion Ψ_N^h is relatively accurate (measured against an approximation from the classical Monte Carlo method) even

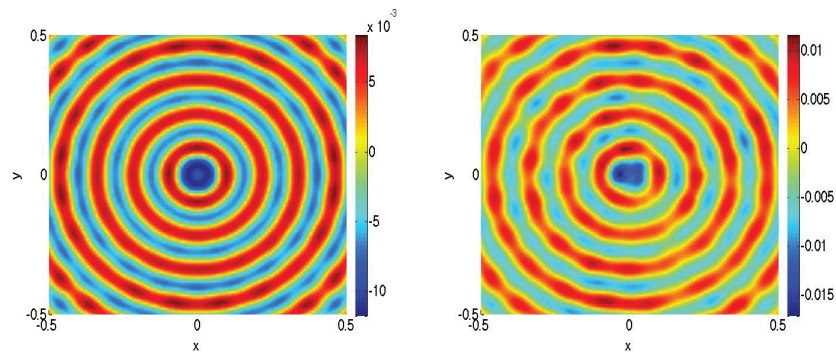


Figure 10. $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.8$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

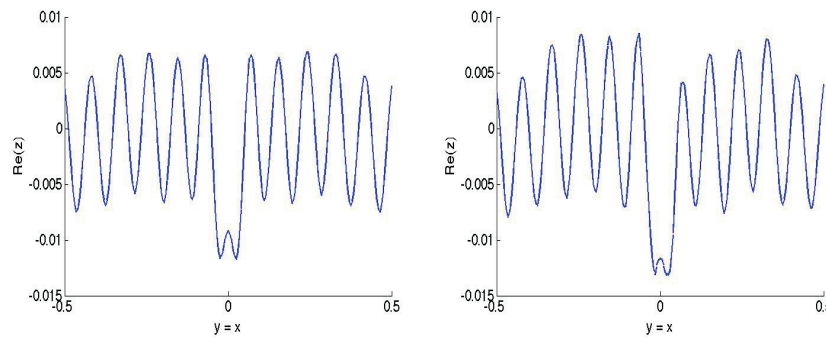


Figure 11. Cross sections of $\text{Re}(\Psi_3^h)$ (left) and $\text{Re}(U_3^h)$ (right) computed for $k = 50$, $h = 1/100$, $\varepsilon = 0.8$, $\eta(\cdot, x) \sim \mathcal{U}[-1, 1]$, and $M = 1000$.

Table 2

L^2 -norm relative error between the multimodes expansion approximation Ψ_3^h and the classical Monte Carlo approximation $\tilde{\Psi}^h$.

ε	0.02	0.1	0.5	0.8
Relative L^2 error	3.0125×10^{-4}	6.0073×10^{-4}	0.2865	1.6979

Table 3

L^2 -norm relative error between the multimodes expansion approximation Ψ_N^h and the classical Monte Carlo approximation $\tilde{\Psi}^h$.

ε	$N = 4$	$N = 5$	$N = 6$	$N = 7$
0.5	0.2866	0.1125	0.1137	0.0554
0.8	1.7036	1.6713	1.6839	1.7887

in cases when ε does not satisfy the constraint set forth in the convergence theory. We also observe that when ε becomes too large, the multimodes expansion no longer agrees with the classical Monte Carlo method.

In the final set of numerical tests our goal is to evaluate the performance of the MCIP-DG

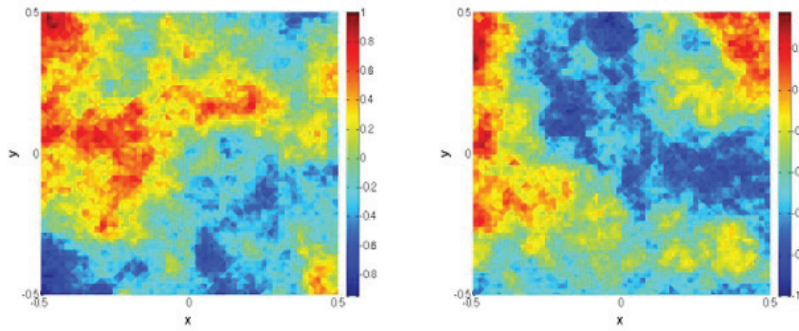


Figure 12. Samples of the random field $\eta(\omega, \cdot)$ generated using an exponential covariance function with covariance length $l = 0.5$ on a triangulation parameterized by $h = 1/50$.

Table 4

L^2 -norm relative error between the multimodes expansion approximation Ψ_3^h and the classical Monte Carlo approximation $\tilde{\Psi}^h$.

ϵ	0.04	0.1	0.5	0.8
Relative L^2 error	5.06×10^{-5}	0.0015	0.3855	1.8485

Table 5

L^2 -norm relative error between the multimodes expansion approximation Ψ_N^h and the classical Monte Carlo approximation $\tilde{\Psi}^h$.

ϵ	$N = 4$	$N = 5$	$N = 6$	$N = 7$
0.5	0.9331	0.6590	1.2208	1.0298
0.8	3.9015	5.1598	13.0311	29.0078

method when the medium refractive index $\alpha(\omega, \cdot)$ is generated using a random field $\eta(\omega, \cdot)$ with a longer correlation length. Here $\eta(\omega, \cdot)$ is a Gaussian random field with an exponential covariance function with correlation length $\ell = 0.5$ (cf. [15]), i.e., the following covariance function is used to generate $\eta(\omega, \cdot)$:

$$C(\mathbf{x}_1, \mathbf{x}_2) = \exp\left(-\frac{\|\mathbf{x}_1 - \mathbf{x}_2\|_2}{0.5}\right).$$

For the numerical simulations $\eta(\omega, \cdot)$ is sampled at the center of each element in the mesh. Sample realizations of $\eta(\omega, \cdot)$ generated in this manner are provided in Figure 12.

We again use $f = \sin(k\alpha(\omega, \cdot)r)/r$, where r is the radial distance from the origin and $g = 0$. We fix $k = 25$, $h = 1/50$, and $M = 1000$. For $\epsilon = 0.04, 0.1, 0.5, 0.8$ the MCIP-DG method is applied using varying total number of modes N . Tables 4 and 5 record the relative error (measured in the L^2 -norm) between the multimodes approximation Ψ_N^h and the classical Monte Carlo approximation $\tilde{\Psi}^h$.

For this test we expect that the MCIP-DG method work well when $\epsilon = 0.04$ and, indeed, in this case the relative error between Ψ_3^h and $\tilde{\Psi}^h$ is small, as shown in Table 4. We also see that when $\epsilon = 0.1$, the relative error between Ψ_3^h and $\tilde{\Psi}^h$ is also small. Table 4 indicates

that in the case of $\varepsilon = 0.5, 0.8$, the relative errors between Ψ_3^h and $\tilde{\Psi}^h$ are no longer small. Similar to the previous tests, for these values of ε we use a larger total number of modes N . From Table 5 we see that increasing the number of modes N in these cases does not improve the relative error, but rather produces larger errors. Comparing these results to the results from the previous tests suggests that a longer correlation length for the random field $\eta(\omega, \cdot)$ will have an adverse effect on the size of ε that will yield an accurate multimodes MCIP-DG approximation. On the other hand, we still observe that the multimodes MCIP-DG method produces an accurate approximation for ε larger than $\frac{1}{k}$. In particular, this is the case when $\varepsilon = 0.1$.

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