# Characterization of the essential spectrum of the Neumann-Poincaré operator in 2D domains with corner via Weyl sequences 

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#### Abstract

The Neumann-Poincaré (NP) operator naturally appears in the context of metamaterials as it may be used to represent the solutions of elliptic transmission problems via potentiel theory. In particular, its spectral properties are closely related to the well-posedness of these PDE's, in the typical case where one considers a bounded inclusion of homogeneous plasmonic metamaterial embedded in a homogeneous background dielectric medium. In a recent work [30], M. Perfekt and M. Putinar have shown that the NP operator of a 2D curvilinear polygon has an essential spectrum, which depends only on the angles of the corners. Their proof is based on quasi-conformal mappings and techniques from complex-analysis. In this work, we characterize the spectrum of the NP operator for a 2D domain with corners in terms of elliptic corner singularity functions, which gives insight on the behavior of generalized eigenmodes.


## 1 Introduction

Plasmonic metamaterials are composite structures, in which some parts are made of media with negative indices. Their fascinating properties of subwavelength confinement and enhancement of electro-magnetic waves have

[^0]drawned considerable interest from the physics and mathematics communities. The progress in the controled production of composites with characteristic features of the order of optical wavelengths contributes to this activity, as it may enable many applications to nano-optical-mechanical systems, cancer therapy, neuro-science, energy and information storage and processsing.

From the mathematical modeling point of view, these studies have also renewed interest in the Neumann-Poincare operator, the integral operator derived from the normal derivative of the single layer potential. Indeed, it proves to be an interesting tool to construct, represent and derive properties of solutions to diffusion-like equations, in situations where the Lax-Milgram theory does not apply, which is typically the case of negative index materials.

The spectral properties of this operator have proved interesting in several contexts [1, 2, 12, 13, 14]. They are particularly relevant to metamaterials, as they are closely related to the existence of surface plasmons, i.e., solutions of the governing PDE (Maxwell, Helmholtz, accoustic equations) which are supported in the vicinity of the interfaces where the coefficients change signs.

To fix ideas, we consider a single inclusion $D$ made of negative index material (typically metals, such as gold or silver at optical frequencies). It is embedded in a homogeneous dielectric background medium and we by denote $\mathcal{K}_{D}^{*}$ the associated NP operator (its precise definition is given in section 2). For particular frequencies, called plasmonic resonant frequencies, an incident wave may excite electrons on the surface of the inclusion into a resonant state, that generates highly oscillating and localized electromagnetic fields. For gold and silver, plasmonic resonances occur when the diameter of the particles is small compared to the wavelength. From the modeling point of view, one may rescale the governing Maxwell or Helmholtz equations, with respect to particule size, and take the limit of the resulting equations to obtain the quasi-static regime, where only the higher-order terms of the original PDE remain [27, 19, 3, 4]. Plasmonic resonances have been investigated via layer potential techniques in [1]-[6].
When $D$ has a smooth boundary (say $\mathcal{C}^{2}$ ) the operator $\mathcal{K}_{D}^{*}$ is compact. Its spectrum is real, contained in the interval $(-1 / 2,1 / 2]$, and consists in a countable number of eigenvalues that accumulates to 0 . In the context of plasmonics, domains with corners present an obvious interest when one attemps to concentrate electro-magnetic fields, and several authors have
considered geometries where the negative index materials are distributed in regions with corners [10, 11, 21, 22]. When $D$ has corners, $\mathcal{K}_{D}^{*}$ is not compact [31]. In a recent work, M.-K. Perfekt and M. Putinar have shown, relying on the relationship betwen complex analysis and potential theory, that the NP operator associated to a planar domain with corners has essential spectrum, which they characterized to be

$$
\sigma_{e s s}\left(\mathcal{K}_{D}^{*}\right)=\left[\lambda_{-}, \lambda_{+}\right], \quad \lambda_{+}=-\lambda_{-}=\frac{1}{2}\left(1-\frac{\alpha}{\pi}\right),
$$

where $\alpha$ is the most acute angle of $D$. See [30, 29].
The objective of our paper is to give an alternative derivation of the essential spectrum of $\mathcal{K}_{D}^{*}$ when $D$ has corners, and to establish a close connection between the fact that $\mathcal{K}_{D}^{*}$ has essential spectrum and the theory of elliptic corner singularities initiated by Kondratiev in the 1970's and developed in many directions. See [25] and also [20, 16, 26] and the many references therein. This theory shows that the solution $u$ to an elliptic scalar equation in a domain $O$ with corners splits as the sum $u=u_{\text {reg }}+u_{\text {sing }}$ of a regular part $u_{\text {reg }} \in H^{2}(O)$ and a singular part $u_{\text {sing }} \in H^{1}(O) \backslash H^{2}(O)$, locally around each corner. Up to a scaling factor, the expression of the latter part, which we call 'singularity function', only depends on the geometry of the corner, and on the nature of the boundary conditions. In the case of a transmission problem, it depends on the angle and on the contrast in material coefficients. Typically, $u_{\text {sing }}$ is a non-trivial solution of a homogeneous problem for the associated operator in the infinite domain obtained by zooming around the vertex of the corner. For a transmission problem in 2D, it has the form

$$
\begin{equation*}
u_{\text {sing }}=C r^{\eta} \varphi(\theta) \tag{1}
\end{equation*}
$$

where $(r, \theta)$ denote the polar coordinates with orgin at the vertex of the corner under consideration. The exponent $\eta$ is the root of a dispersion relation, and $\varphi$ is a smooth function (or piecewise smooth in the case of a transmission problem).

This paper is organized in the following way. Section 2 of the paper describes the setting and notations and reviews useful facts about the NP operator. In Section 3, we study how elliptic corner singularity functions depend on the conductivity contrast. In the very interesting papers [9, 10, 11, it is shown that functions of the form (17) only exist when the conductivity contrast $\lambda$ lies outside a critical interval $\left[\lambda_{-}, \lambda_{+}\right]$. When $\lambda \in\left[\lambda_{-}, \lambda_{+}\right]$, the elliptic corner singularity functions still have the form $u_{\text {sing }}$ but their expression involves
a complex exponent $\eta$. In [11], the use of the Mellin transform converts the search of these singular functions to that of propagative mode in an infinite wave-guide. These functions are called plasmonic black-hole waves, reflecting the fact that they are not in the energy space $H^{1}(\Omega)$. In Section 4, we show that the critical interval is contained in the essential spectrum $\sigma_{\text {ess }}\left(\mathcal{K}_{D}^{*}\right)$, by generating singular Weyl sequences [8] using the singularity functions. In Section 5, the reverse inclusion is proved. In particular, we use a construction inspired by [28] to transform, around the vertex of the corner, the PDE with sign changing conditions into a system of PDE's defined in the inhomogeneity only, that satisfies complementing boundary conditions in the sense of Agmon, Douglis and Nirenberg, and for which we prove well-posedness.

## 2 The Neumann-Poincaré operator and the Poincaré variational operator

Throughout the text, $\Omega \subset \mathbf{R}^{2}$ denotes a bounded open set with smooth boundary, that strictly contains a connected inclusion $D$. We assume that $\partial D$ is smooth, except for one corner point, of angle $\alpha, 0<\alpha<P i$, located at the origin. We assume that for some $R_{0}>0$,

$$
\begin{equation*}
D \cap B_{R_{0}}=\left\{x=(r \cos (\theta), r \sin (\theta)), 0 \leq r<R_{0},|\theta|<\alpha / 2\right\}, \tag{2}
\end{equation*}
$$

where, for any $\rho>0, B_{\rho}$ denotes the ball of radius $\rho$ centered at 0 . The space $H_{0}^{1}(\Omega)$ is equipped with the following inner product and associated norm

$$
<u, v>_{H_{0}^{1}}=\int_{\Omega} \nabla u \cdot \nabla v d x, \quad\|u\|_{H_{0}^{1}}=\left(\int_{\Omega}|\nabla u|^{2} d x\right)^{1 / 2} .
$$

Our work concerns the following diffusion equation: given a function $f \in L^{2}(\Omega)$, we seek $u$ such that

$$
\left\{\begin{array}{ccc}
-\operatorname{div}(a(x) \nabla u(x)) & =f \text { in } \Omega,  \tag{3}\\
u(x) & =0 & \text { on } \partial \Omega,
\end{array}\right.
$$

where the conductivity $a$ is piecewise constant

$$
a(x)=\left\{\begin{array}{cc}
k \in \mathbf{C} & x \in D  \tag{4}\\
1 & x \in \Omega \backslash \bar{D}
\end{array}\right.
$$

It is well known that when $k$ is strictly positive, or when $k \in \mathbf{C}$ and $\operatorname{Im}(k) \neq 0$, this problem has a unique solution in $H^{1}(\Omega)$, and that

$$
\|u\|_{H_{0}^{1}} \leq C(k)\|f\|_{L^{2}},
$$

for some constant $C(k)>0$ that depends on $k$.
Let $P(x, y)$ denote the Poisson kernel associated to $\Omega$, defined by

$$
P(x, y)=G(x, y)+R_{x}(y), \quad x, y \in \Omega
$$

where $G(x, y)$ denotes the free space Green function

$$
G(x, y)=\frac{1}{2 \pi} \ln |x-y|,
$$

and where $R_{x}(y)$ is the smooth solution to

$$
\left\{\begin{array}{cccl}
\Delta_{y} R_{x}(y) & = & 0 & y \in \Omega, \\
R_{x}(y) & = & -G(x, y) & y \in \partial \Omega .
\end{array}\right.
$$

With the Poisson kernel, we define the single layer potentials $\mathcal{S}_{D} \varphi \in L^{2}(\partial D)$ of a function $\varphi \in L^{2}(\partial D)$ by

$$
\mathcal{S}_{D} \varphi(x)=\int_{\partial D} P(x, y) \varphi(y) d s(y), \quad x \in D \cup(\Omega \backslash \bar{D}) .
$$

It is well known [18, 31 that $\mathcal{S}_{D} \varphi$ is harmonic in $D$ and in $\Omega \backslash \bar{D}$, continous in $\bar{\Omega}$, and that its normal derivatives satisfy the Plemelj jump conditions

$$
\begin{equation*}
\left.\frac{\partial \mathcal{S}_{D} \varphi}{\partial \nu}\right|^{ \pm}(x)=\left( \pm \frac{1}{2} I+\mathcal{K}_{D}^{*}\right) \varphi(x), \quad x \in \partial D . \tag{5}
\end{equation*}
$$

where $\mathcal{K}_{D}^{*}$ is the Neumann-Poincaré operator, defined by

$$
\mathcal{K}_{D}^{*} \varphi(x)=\int_{\partial D} \frac{\partial P}{\partial \nu_{y}}(x, y) \varphi(y) d s(y) .
$$

It is shown in 15 that this definition makes sense for Lipschitz domains, and in that case, the operator $\mathcal{K}_{D}^{*}$ is continuous from $L^{2}(\partial D) \rightarrow L^{2}(\partial D)$, which extends as an operator $H^{-1 / 2}(\partial D) \rightarrow H^{1 / 2}(\partial D)$.
The solution $u$ to (3) can then be represented in the form

$$
\begin{equation*}
u(x)=\mathcal{S}_{D} \varphi(x)+H(x), \tag{6}
\end{equation*}
$$

where the harmonic part is given by

$$
H(x)=\int_{\Omega} P(x, y) f(y) d s(y)
$$

The jump conditions (5), constrain the layer potential $\varphi \in H^{-1 / 2}(\partial D)$ to satisfy the integral equation

$$
\left(\lambda I-\mathcal{K}_{D}^{*}\right) \varphi(x)=\partial_{\nu} H_{\mid \partial D}(x), \quad x \in \partial D
$$

We also introduce the Poincaré variational operator $T_{D}: H_{0}^{1}(\Omega) \rightarrow H_{0}^{1}(\Omega)$, defined for $u \in H_{0}^{1}(\Omega)$ by

$$
\begin{equation*}
\forall v \in H_{0}^{1}(\Omega), \quad \int_{\Omega} \nabla T_{D} u \cdot \nabla v d x=\int_{D} \nabla u \cdot \nabla v d x \tag{7}
\end{equation*}
$$

Some of its properties are described in the following proposition (see [12] for a proof).

Proposition 1. The operator $T_{D}$ is bounded, selfadjoint, and satisfies $\left\|T_{D}\right\|=1$. Moreover,
(i) Its spectrum $\sigma\left(T_{D}\right)$ is contained in the interval $[0,1]$.
(ii) Its kernel, the eigenspace associated to $\beta=0$, is

$$
\operatorname{Ker}\left(T_{D}\right)=\left\{u \in H_{0}^{1}(\Omega), u=\text { const on } D\right\}
$$

(iii) $1 \in \sigma\left(T_{D}\right)$ and the associated eigenspace is

$$
\operatorname{Ker}\left(I-T_{D}\right)=\left\{u \in H_{0}^{1}(\Omega), u=0 \text { in } \Omega \backslash \bar{D}\right\}
$$

(and thus, can be identified with $H_{0}^{1}(D)$ ).
(iv) The space $H_{0}^{1}(\Omega)$ decomposes as

$$
H_{0}^{1}(\Omega)=\operatorname{Ker}\left(T_{D}\right) \oplus \operatorname{Ker}\left(I-T_{D}\right) \oplus \mathcal{H}
$$

where $\mathcal{H}$ is the closed subspace defined by

$$
\mathcal{H}=\left\{u \in H_{0}^{1}(\Omega), \Delta u=0 \text { in } D \cup(\Omega \backslash \bar{D}), \int_{\partial D} \frac{\partial u^{+}}{\partial \nu} d s=0\right\}
$$

The space $\mathcal{H}_{s}=\mathcal{H} \oplus \operatorname{Ker}\left(T_{D}\right)$ is the space of single layer potentials. It is isomorphic to

$$
H_{0}^{-1 / 2}(\partial D)=\left\{\varphi \in H^{1 / 2}(\partial D),<\varphi, 1>_{H^{-1 / 2}, H^{1 / 2}}=0\right\}
$$

This latter space is equipped with the inner product

$$
\begin{equation*}
<\varphi, \psi>_{S}=-\int_{\partial D} \varphi S \psi d \sigma \tag{8}
\end{equation*}
$$

for which the operator $\mathcal{K}_{D}^{*}: H_{0}^{-1 / 2}(\partial D) \rightarrow H_{0}^{-1 / 2}(\partial D)$ is self-ajdjoint as a result of the Calderón identity [24]. We denote by $\|\cdot\|_{S}$ the associated norm. In particular, if $u, v \in \mathcal{H}_{s}$ are such that $u=S_{D} \varphi, v=S_{D} \psi$, then the jump conditions (5) and integration by parts show that

$$
\begin{equation*}
\int_{\Omega} \nabla u \cdot \nabla v=<\varphi, \psi>_{S} \tag{9}
\end{equation*}
$$

When the domain $D$ has a $\mathcal{C}^{2}$ boundary, the Poincaré-Neumann operator $\mathcal{K}_{D}^{*}: H_{0}^{-1 / 2}(\partial D) \rightarrow H_{0}^{-1 / 2}(\partial D)$ is compact. Its spectrum $\sigma\left(\mathcal{K}_{D}^{*}\right)$ is contained in $[-1 / 2,1 / 2]$, and consists of a sequence of real eigenvalues that accumulates to 0 . In this case, $\sigma\left(\mathcal{K}_{D}^{*}\right)$ is directly related to $\sigma\left(T_{D}\right)$. Indeed, if $u \in H_{0}^{1}(\Omega)$ and $\beta \in \mathbf{R}, \beta \neq 1$, satisfy $T_{D} u=\beta u$, it follows from (7) that

$$
\forall v \in H_{0}^{1}(\Omega), \quad \beta \int_{\Omega \backslash D} \nabla u \cdot \nabla v d x+(\beta-1) \int_{D} \nabla u \cdot \nabla v d x=0
$$

so that $u$ is a non-zero solution to

$$
\left\{\begin{array}{cll}
\operatorname{div}(a(x) \nabla u(x)) & = & \text { in } \Omega,  \tag{10}\\
u(x) & = & 0
\end{array} \text { on } \partial \Omega,\right.
$$

where the conductivity $a$ equals $\beta$ in $\Omega \backslash \bar{D}$ and $(\beta-1)$ in $D$. Expressing $u$ in the form $u=\mathcal{S}_{D} \varphi$ yields yields the integral equation

$$
\left(\lambda I-\mathcal{K}_{D}^{*}\right) \varphi(x)=0, \quad x \in \partial D
$$

where $\lambda=1 / 2-\beta$ is thus an eigenvalue of $\mathcal{K}_{D}^{*}$. It follows that

$$
\sigma\left(T_{D}\right)=\left(1 / 2-\sigma\left(\mathcal{K}_{D}^{*}\right)\right) \cup\{0,1\}
$$

As recalled above, when $D$ is a domain with corners, $\sigma\left(\mathcal{K}_{D}^{*}\right)$ contains an interval of essential spectrum [30]. We have

Proposition 2. The essential spectra of $T_{D}$ and $\mathcal{K}_{D}^{*}$ are related by $\sigma_{\text {ess }}\left(T_{D}\right)=$ $1 / 2-\sigma_{\text {ess }}\left(\mathcal{K}_{D}^{*}\right)$.

Proof: Let $\lambda \in \sigma_{\text {ess }}\left(\mathcal{K}_{D}^{*}\right)$. By definition, there exists a singular Weyl sequence, i.e., a sequence of functions $\left(\varphi_{\varepsilon}\right) \subset H_{0}^{-1 / 2}$ such that

$$
\left\{\begin{array}{cl}
\left(\lambda I-\mathcal{K}_{D}^{*}\right) \varphi_{\varepsilon} & \rightarrow 0 \\
\left\|\varphi_{\varepsilon}\right\|_{S} & =1, \\
\varphi_{\varepsilon} & \longrightarrow 0
\end{array} \quad \text { strongly in } H_{0}^{-1 / 2},\right.
$$

Let $\beta=1 / 2-\lambda$ and $u_{\varepsilon}=S_{D} \varphi_{\varepsilon} \in \mathcal{H}_{S}$. Let $v \in \mathcal{H}_{S}$ so that $v=S_{D} \psi$ for some $\psi \in H_{0}^{-1 / 2}(\partial D)$. It follows from (9) that

$$
\int_{\Omega} \nabla u_{\varepsilon} \cdot \nabla v=<\varphi_{\varepsilon}, \psi>\rightarrow 0
$$

This equality also holds for $v \in \operatorname{Ker}\left(I-T_{D}\right)$ since this subspace is orthogonal to $\mathcal{H}_{S}$, and thus

$$
\begin{equation*}
u_{\varepsilon} \rightharpoonup 0 \text { weakly in } H_{0}^{1}(\Omega) \tag{11}
\end{equation*}
$$

Additionally, invoking (9) again, we see that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}=<\varphi_{\varepsilon}, \varphi_{\varepsilon}>_{S}=1 \tag{12}
\end{equation*}
$$

Finally, we compute, for $v=S_{D} \psi \in \mathcal{H}_{S}$,

$$
\begin{aligned}
\int_{\Omega} \nabla\left(\left(\beta I-T_{D}\right) u_{\varepsilon}\right) \cdot \nabla v & =\int_{\Omega} \beta \nabla u_{\varepsilon} \cdot \nabla v-\int_{D} \nabla u_{\varepsilon} \cdot \nabla v \\
& =\int_{\Omega \backslash D} \beta \nabla u_{\varepsilon} \cdot \nabla v+\int_{D}(\beta-1) \nabla u_{\varepsilon} \cdot \nabla v \\
& =-\left.\beta \int_{\partial D} \partial_{\nu} u_{\varepsilon}\right|^{+} v+\left.(\beta-1) \int_{\partial D} \partial_{\nu} u_{\varepsilon}\right|^{-} v .
\end{aligned}
$$

Inserting (5) in place of the normal derivatives of $u_{\varepsilon}$ we see that

$$
\begin{aligned}
\int_{\Omega} \nabla\left(\left(\beta I-T_{D}\right) u_{\varepsilon}\right) \cdot \nabla v & =<\left(\lambda I-\mathcal{K}_{D}^{*}\right) \varphi_{\varepsilon}, \psi>_{S} \\
& \leq\left\|\left(\lambda I-\mathcal{K}_{D}^{*}\right) \varphi_{\varepsilon}\right\|_{S}\|\psi\|_{S}
\end{aligned}
$$

It follows that

$$
\begin{equation*}
\left\|\left(\beta I-T_{D}\right) u_{\varepsilon}\right\|_{H_{1}} \leq\left\|\left(\lambda I-\mathcal{K}_{D}^{*}\right) \varphi_{\varepsilon}\right\|_{S} \rightarrow 0 \tag{13}
\end{equation*}
$$

we conclude from (11, 13) that $u_{\varepsilon}$ is a singular Weyl sequence associated to $\beta$, so that $\beta \in \sigma_{e s s}\left(T_{D}\right)$. The same argument proves the reverse inclusion $\sigma_{\text {ess }}\left(T_{D}\right) \subset\left(1 / 2-\sigma_{\text {ess }}\left(\mathcal{K}_{D}^{*}\right)\right)$.

## 3 Corner singularity functions

Elliptic corner singularities have been the subject of much research since the pionneering works of Kondratiev [25], Grisvard [20] (see also [16, 26]). Essentially, the theory focuses on the regularity of solutions to elliptic PDEs near a corner of the domain, or in the case of a transmission problem such as (3), near a corner of the interface between several phases. The following is a typical statement:

Theorem 1. Let $k>0$. The solution $u \in H_{0}^{1}(\Omega)$ to (3) decomposes as

$$
u=u_{\text {sing }}+u_{r e g},
$$

where $u_{\text {reg }} \in H^{2}(\Omega)$ and where $u_{\text {sing }}$ has the form

$$
\begin{equation*}
u_{\text {sing }}(x)=r^{\eta} \varphi(\theta) \zeta(x), \quad x \in \Omega . \tag{14}
\end{equation*}
$$

Here $x=(r \cos (\theta), r \sin (\theta))$ in polar coordinates, $\zeta$ is a smooth cut-off function, such that, for some $s>0$

$$
\zeta(x)= \begin{cases}1 & |x|<s \\ 0 & |x|>2 s .\end{cases}
$$

Moreover, for some constant $C=C(\alpha, k)$, the following estimate holds

$$
\begin{equation*}
\left\|u_{s i n g}\right\|_{H^{1}(\Omega)}+\left\|u_{r e g}\right\|_{H^{2}(\Omega)} \leq C\left(\|u\|_{H^{1}(\Omega)}+\|f\|_{L^{2}(\Omega)}\right) . \tag{15}
\end{equation*}
$$

These results have been derived only in the case of elliptic media, i.e., when $k>0$. In the context of plasmonic metamaterials, it is natural to try to extend them to complex values of $k$. To our best knowledge, the first steps in this direction have been obtained in [9, 10] and concern the existence of singularity functions of the form (14).

### 3.1 Regular corner singularity functions

In this paragraph, we investigate whether one can define singular functions such as (14) when $k$ may also take negative values. More precisely, we seek $H_{l o c}^{1}\left(\mathbf{R}^{2}\right)$ solutions to

$$
\begin{equation*}
\operatorname{div}(a(x) \nabla u(x))=0 \quad \text { in } \mathbf{R}^{2}, \tag{16}
\end{equation*}
$$

of the form

$$
\begin{equation*}
u(x)=r^{\eta} \varphi(\theta), \eta \in \mathbf{R}, \tag{17}
\end{equation*}
$$

when the conductivity $a(x)$ is defined in the whole of $\mathbf{R}^{2}$ by

$$
a(x)=\left\{\begin{array}{cl}
k & |\theta|<\alpha / 2  \tag{18}\\
1 & \text { otherwise }
\end{array}\right.
$$

Since we are only interested in singular solutions which belong to $H^{1}(\Omega) \backslash$ $H^{2}(\Omega)$, we may restrict $\eta$ to lie in $(0,1)$. As $u$ is harmonic in each sector $|\theta|<\alpha / 2$ and $\alpha / 2<\theta<2 \pi-\alpha / 2$, it follows that $\varphi$ has the form
$\varphi= \begin{cases}a_{1} \cos (\eta(\theta+\alpha / 2))+b_{1} \sin (\eta(\theta+\alpha / 2)) & \text { if }-\alpha / 2<\theta<\alpha / 2, \\ a_{2} \cos (\eta(\theta+\alpha / 2))+b_{2} \sin (\eta(\theta+\alpha / 2)) & \text { if } \alpha / 2<\theta<2 \pi-\alpha / 2\end{cases}$
for some $a_{i}, b_{i}, i=1,2$. Expressing the continuity of $u$ and of $a(x) \partial_{\nu} u$ across the interfaces, shows that a non-trivial solution exits if and only if the following dispersion relation is satisfied

$$
\operatorname{det}\left(\begin{array}{cccc}
1 & 0 & -\cos (2 \pi \eta) & -\sin (2 \pi \eta) \\
\cos (\alpha \eta) & \sin (\alpha \eta) & -\cos (\alpha \eta) & -\sin (\alpha \eta) \\
0 & k & \sin (2 \pi \eta) & -\cos (2 \pi \eta) \\
-k \sin (\alpha \eta) & k \cos (\alpha \eta) & -\sin (\alpha \eta) & -\cos (\alpha \eta)
\end{array}\right)=0,
$$

which, after elementary manipulations, can be rewritten in the form

$$
\begin{equation*}
\frac{2 k}{k^{2}+1}=\frac{\sin (\alpha \eta) \sin ((2 \pi-\alpha) \eta)}{1-\cos (\alpha \eta) \cos ((2 \pi-\alpha) \eta)}=: F(\eta, \alpha) . \tag{20}
\end{equation*}
$$

A Taylor expansion around the values $\eta=0$ shows that $F(\cdot, \alpha)$ can be extended by continuity to a function defined on the whole of $[0,1]$ by setting

$$
F(0, \alpha)=\frac{-2 \alpha(2 \pi-\alpha)}{\alpha^{2}+(2 \pi-\alpha)^{2}} .
$$

By solving

$$
\frac{2 k}{k^{2}+1}=F(0, \alpha),
$$

we obtain two solutions

$$
\begin{equation*}
k_{+}=\frac{-(2 \pi-\alpha)}{\alpha}, \quad k_{-}=\frac{-\alpha}{2 \pi-\alpha} . \tag{21}
\end{equation*}
$$

Additionnally, it is easy to check that $|F(\eta, \alpha)| \leq 1$ and

$$
\partial_{\eta} F=\frac{\cos ((2 \pi-\alpha) \eta)-\cos (\alpha \eta))[a \sin ((2 \pi-\alpha) \eta)-(2 \pi-\alpha) \sin (\alpha \eta)]}{[1-\cos (\alpha \eta) \cos ((2 \pi-\alpha) \eta)]^{2}} .
$$

We show below that $F(\cdot, \alpha)$ is thus strictly increasing, and note that $\partial_{\eta} F(0, \alpha)=$ $\partial_{\eta} F(1, \alpha)=0$.

Lemma 1. For any $0<\alpha<\pi$ and $0 \leq \eta \leq 1$, the following inequalities hold

$$
\begin{align*}
\cos ((2 \pi-\alpha) \eta)-\cos (\alpha \eta) & <0  \tag{22}\\
a \sin ((2 \pi-\alpha) \eta)-(2 \pi-\alpha) \sin (\alpha \eta) & <0 .
\end{align*}
$$

Proof: To prove the first inequality, we first note that $\alpha<(2 \pi-\alpha)$ so that $\alpha \eta<(2 \pi-\alpha) \eta$. If $(2 \pi-\alpha) \eta \leq \pi$, then 22 ) follows from the monotonicity of the cosine function on $[0, \pi]$. If $(2 \pi-\alpha) \eta>\pi$, then

$$
\cos ((2 \pi-\alpha) \eta)=\cos (\pi-\beta), \quad \text { with }(2 \pi-\alpha) \eta=: \pi+\beta
$$

Noticing that

$$
\alpha \eta \leq \alpha \eta+2 \pi(1-\eta)=\pi-\beta<\pi,
$$

we infer that $\cos (\pi-\beta)<\cos (\alpha \eta)$, which yields the result.
The second inequality follows from the fact that

$$
\begin{aligned}
& \partial_{\eta}[a \sin ((2 \pi-\alpha) \eta)-(2 \pi-\alpha) \sin (\alpha \eta)] \\
& \quad=\alpha(2 \pi-\alpha)[\cos ((2 \pi-\alpha) \eta)-\cos (\alpha \eta)]
\end{aligned}
$$

which according to 22 is negative.

As a consequence of (20), we obtain
Proposition 3. Singular solutions in $H_{l o c}^{1}\left(\mathbf{R}^{2}\right)$ of the form (17) exists for the equation (16) only when $k \in\left(-\infty, k_{+}\right) \cup\left(k_{-},+\infty\right)$, see Figure 3.1. In terms of the contrast $\lambda=\frac{k+1}{2(k-1)}$ this condition is equivalent to

$$
\lambda \notin\left[\lambda_{-}, \lambda_{+}\right]:=\left[-\frac{1}{2}\left(1-\frac{\alpha}{\pi}\right), \frac{1}{2}\left(1-\frac{\alpha}{\pi}\right)\right] .
$$

In other words, singular solutions of the form (17) only exist when $\lambda=$ $\frac{k+1}{2(k-1)}$ is not in $\sigma_{\text {ess }}\left(\mathcal{K}_{D}^{*}\right)$.



Figure 1: Left: Plot of the function $k \rightarrow 2 k /\left(k^{2}+1\right)$. Right: Plot of $\eta \rightarrow F(\eta, \alpha)$ (blue), and of $\xi \rightarrow \tilde{F}(\xi, \alpha)$ (green), for $\alpha=\pi / 2$. The dotted line indicates the value of $2 k /\left(k^{2}+1\right)$ below which the dispersion relation has no solution $\eta \in \mathbf{R}$.

### 3.2 Singular corner singularity functions

We now construct local singular solutions when $k \in\left[k_{+}, k_{-}\right]$. By this we mean functions which satisfy the PDE (16), but which may only be in $H_{l o c}^{1}\left(\mathbf{R}^{2} \backslash\{0\}\right)$. To this end, we seek $u(x)=r^{\eta} \varphi(\theta)$, with $\varphi$ in the form 19), but assume now that $\eta \in \mathbf{C}$. The same algebra leads to the same dispersion relation (20). In particular if we restrict $\eta$ to be a pure imaginary number, $\eta=i \xi$, this relation takes the form

$$
\frac{2 k}{k^{2}+1}=\frac{\sinh (\alpha \xi) \sinh ((2 \pi-\alpha) \xi)}{1-\cosh (\alpha \xi) \cosh ((2 \pi-\alpha) \xi)}=: \tilde{F}(\xi, \alpha) .
$$

It is easy to check that the function $\xi \rightarrow \tilde{F}(\xi, \alpha)$ can be extended by continuity at $\xi=0$ by setting

$$
\tilde{F}(0, \alpha)=\frac{2 \alpha(2 \pi-\alpha)}{\alpha^{2}+(2 \pi-\alpha)^{2}}=\frac{2 k_{ \pm}}{k_{ \pm}^{2}+1}
$$

(and we note that $\tilde{F}(0, \alpha)=F(0, \alpha))$. In addition, we compute
$\partial_{\xi} \tilde{F}=\frac{(\cosh ((2 \pi-\alpha) \xi)-\cosh (\alpha \xi))[(2 \pi-\alpha) \sinh (\alpha \xi)-\alpha \sinh ((2 \pi-\alpha) \xi)]}{[1-\cosh (\alpha \xi) \cosh ((2 \pi-\alpha) \xi)]^{2}}$.
Just as in Proposition 1, one can show that for any $\xi>0$ and $0<\alpha<\pi$, the product of the two factors in the above numerator is negative, so that $\tilde{F}(\cdot, \alpha)$ is strictly decreasing on $\mathbf{R}^{+}$and its range is equal to $[F(0, \alpha),-1)$, see Figure 3.1. We also note that $\lim _{\eta \rightarrow \infty} \tilde{F}(\eta, \alpha)=-1$, so that the value -1 (which corresponds to $k=-1$ ) is never attained. Summarizing, we have shown that

Proposition 4. For any value of $\lambda \in\left(\lambda_{-}, \lambda_{+}\right), \lambda \neq 0$, there exists $\xi>0$ and a function $u(x)=r^{i \xi} \varphi(\theta)$, which is a local solution to $\operatorname{div}(a(x) \nabla u(x))=0$, where $a$ is defined by (18), with $\lambda=\frac{k+1}{2(k-1)}$.

## 4 Construction of singular Weyl sequences

In this section we prove
Theorem 2. The set $\left[\lambda_{-}, \lambda_{+}\right]$is contained in $\sigma_{\text {ess }}\left(T_{D}\right)$.

## Proof:

Since $\sigma_{\text {ess }}\left(\mathcal{K}_{D}^{*}\right)$ is a closed set, it is sufficient to show that $\left(\lambda_{-}, \lambda_{+}\right) \backslash\{0\} \subset$ $\sigma_{\text {ess }}\left(T_{D}\right)$. We proceed as follows: We consider $\lambda \in\left(\lambda_{-}, \lambda_{+}\right), \lambda \neq 0$, and show that $\beta=1 / 2-\lambda \in \sigma_{\text {ess }}\left(T_{D}\right)$ by constructing a singular Weyl sequence, i.e., a sequence of functions $u_{\varepsilon} \in H_{0}^{1}(\Omega)$, such that

$$
\left\{\begin{array}{ccc}
\left\|u_{\varepsilon}\right\|_{H_{0}^{1}} & = & 1,  \tag{23}\\
\left(\beta I-T_{D}\right) u_{\varepsilon} & \rightarrow & 0 \\
\text { strongly in } H_{0}^{1}(\Omega), \\
u_{\varepsilon} & & 0
\end{array} \quad \text { weakly in } H_{0}^{1}(\Omega) . ~ \$\right.
$$

According to Proposition4, there exists $\xi>0$ and coefficients $a_{1}, b_{2}, a_{2}, b_{2} \in$ $\mathbf{C}$, not all equal to 0 , such that the function

$$
u(x)=\operatorname{Re}\left(r^{i \xi}\right) \phi(\theta)=\left\{\begin{array}{c}
\operatorname{Re}\left(r^{i \xi}\right)\left[a_{1} \cos (i \xi(\theta+\alpha / 2))+b_{1} \sin (i \xi(\theta+\alpha / 2))\right]  \tag{24}\\
\text { if }-\alpha / 2<\theta<\alpha / 2, \\
\operatorname{Re}\left(r^{i \xi}\right)\left[a_{2} \cos (i \xi(\theta+\alpha / 2))+b_{2} \sin (i \xi(\theta+\alpha / 2))\right] \\
\text { otherwise },
\end{array}\right.
$$

is harmonic in $\left(D \cap B_{R_{0}}\right) \backslash\{0\}$ and in $\left((\Omega \backslash \bar{D}) \cap B_{R_{0}}\right) \backslash\{0\}$, and satisfies the transmission conditions at the interfaces $\theta= \pm \alpha / 2$.
Let $r_{0}<R_{0} / 2$ and let $\chi_{1}, \chi_{2}: \mathbf{R}^{+} \rightarrow[0,1]$ denote two smooth cut-off functions, such that for some constant $C>0$

$$
\begin{cases}\chi_{1}(s)=0 \quad|s| \leq 1, & \chi_{2}(s)=0 \quad|s| \geq 2 r_{0} \\ \chi_{1}(s)=1 \quad|s| \geq 2, & \chi_{2}(s)=1 \\ \left|\chi_{1}^{\prime}(s)\right| \leq C, & \left|\chi_{2}^{\prime}(s)\right| \leq C\end{cases}
$$

We set $\chi_{1}^{\varepsilon}(r)=\chi_{1}(r / \varepsilon)$, and define

$$
\begin{equation*}
u_{\varepsilon}(x)=s_{\varepsilon} \chi_{1}^{\varepsilon}(r) \chi_{2}(r) u(x), \quad x \in \Omega \tag{25}
\end{equation*}
$$

The function $u$ is not in $H^{1}$ as its gradient blows up like $r^{-1}$ near the corner, consequently

$$
m_{\varepsilon}:=\int_{\varepsilon}^{r_{0}} \int_{0}^{2 \pi}|\nabla u(x)|^{2} r d r d \theta \rightarrow \infty \quad \text { as } \varepsilon \rightarrow 0
$$

We choose $s_{\varepsilon}$ in 25 so that $\left\|u_{\varepsilon}\right\|_{H_{0}^{1}}=1$, in other words

$$
\begin{aligned}
s_{\varepsilon}^{-2} & =\int_{\varepsilon}^{2 \varepsilon} \int_{0}^{2 \pi}\left|u \nabla \chi_{1}^{\varepsilon}+\chi_{1}^{\varepsilon} \nabla u\right|^{2}+m_{\varepsilon}+\int_{r_{0}}^{2 r_{0}} \int_{0}^{2 \pi}\left|u \nabla \chi_{2}+\chi_{2} \nabla u\right|^{2} \\
& =: J_{1}+m_{\varepsilon}+J_{2} .
\end{aligned}
$$

The term $J_{2}$ is independent of $\varepsilon$ and is $O(1)$, and in particular

$$
J_{2}=o\left(m_{\varepsilon}\right) \quad \text { as } \varepsilon \rightarrow 0
$$

The other term can be estimated as follows

$$
\begin{align*}
J_{1}= & \int_{\varepsilon}^{2 \varepsilon} \int_{0}^{2 \pi}\left|\frac{r^{i \xi}+r^{-i \xi}}{2} \varphi(\theta) \chi_{1}^{\prime}(r / \varepsilon) / \varepsilon+i \xi \frac{r^{i \xi-1}-r^{-i \xi-1}}{2} \varphi(\theta) \chi_{1}(r / \varepsilon)\right|^{2} \\
& +\left|\frac{r^{i \xi-1}-r^{-i \xi-1}}{2} \varphi^{\prime}(\theta) \chi_{1}(r / \varepsilon)\right|^{2} r d r d \theta  \tag{26}\\
\leq & C \int_{0}^{2 \pi}\left(|\varphi(\theta)|^{2}+\left|\varphi^{\prime}(\theta)\right|^{2}\right) d \theta \int_{\varepsilon}^{2 \varepsilon}\left(\left\|\chi_{1}^{\prime}\right\|_{\infty}^{2} / \varepsilon^{2}+r^{-2}\left\|\chi_{1}\right\|_{\infty}\right) r d r \\
\leq & C \int_{0}^{2 \pi}\left(|\varphi(\theta)|^{2}+\left|\varphi^{\prime}(\theta)\right|^{2}\right) d \theta\left(3 / 2\left\|\chi_{1}^{\prime}\right\|_{\infty}^{2}+(\ln (2 \varepsilon)-\ln (\varepsilon))\left\|\chi_{1}\right\|_{\infty}\right) .
\end{align*}
$$

Since $\varphi$ is independent of $\varepsilon$, we see that

$$
J_{1}=O(1)=o\left(m_{\varepsilon}\right), \quad \text { as } \varepsilon \rightarrow 0,
$$

and so $s_{\varepsilon} \sim m_{\varepsilon}^{-1 / 2} \rightarrow 0$.
We next show that $\left\|\left(\beta I-T_{D}\right) u_{\varepsilon}\right\|_{H^{1}} \rightarrow 0$. Indeed, let $v \in H_{0}^{1}(\Omega)$ and consider

$$
\begin{aligned}
J & =\int_{\Omega} \nabla\left(\beta I-T_{D}\right) u_{\varepsilon} \cdot \nabla v \\
& =\int_{\Omega \backslash D} \beta \nabla u_{\varepsilon} \cdot \nabla v+\int_{D}(\beta-1) \nabla u_{\varepsilon} \cdot \nabla v,
\end{aligned}
$$

in view of the definition of $T_{D}$. Inserting the expression (25) of $u_{\varepsilon}$, we see that

$$
\begin{aligned}
J= & s_{\varepsilon} \int_{\Omega \backslash D} \beta \nabla u \cdot \nabla\left(\chi_{1}^{\varepsilon} \chi_{2} v\right)+s_{\varepsilon} \int_{D}(\beta-1) \nabla u \cdot \nabla\left(\chi_{1}^{\varepsilon} \chi_{2} v\right) \\
& +s_{\varepsilon} \int_{\Omega \backslash D} \beta u \nabla\left(\chi_{1}^{\varepsilon} \chi_{2}\right) \cdot \nabla v+s_{\varepsilon} \int_{D}(\beta-1) u \nabla\left(\chi_{1}^{\varepsilon} \chi_{2}\right) \cdot \nabla v \\
& -s_{\varepsilon} \int_{\Omega \backslash D} \beta \nabla u \cdot v \nabla\left(\chi_{1}^{\varepsilon} \chi_{2}\right)-s_{\varepsilon} \int_{D}(\beta-1) \nabla u \cdot v \nabla\left(\chi_{1}^{\varepsilon} \chi_{2}\right) .
\end{aligned}
$$

Since $u$ is a local solution to (16), the sum of the first 2 integrals vanishes, and we remain with

$$
\begin{align*}
J= & \left(s_{\varepsilon} \int_{\Omega} a u \nabla\left(\chi_{1}^{\varepsilon} \chi_{2}\right) \cdot \nabla v+s_{\varepsilon} \int_{\Omega \cap\left(B_{\left.2 r_{0} \backslash B_{r_{0}}\right)} a u \nabla\left(\chi_{2}\right) \cdot \nabla v\right)}+s_{\varepsilon} \int_{\Omega \cap\left(B_{2 \varepsilon} \backslash B_{\varepsilon}\right)} a v \nabla\left(\chi_{1}^{\varepsilon}\right) \cdot \nabla u=: s_{\varepsilon}\left(J_{3}+J_{4}\right) .\right.
\end{align*}
$$

where $a=\beta$ in $\Omega \backslash \bar{D}$ and $a=\beta-1$ in $D$. The Cauchy-Schwarz inequality allows us to estimate the first two terms on the right-hand side by

$$
\begin{align*}
\left|J_{3}\right| \leq & C s^{\varepsilon}\|v\|_{H^{1}}\left\{\int_{0}^{2 \varepsilon} \int_{0}^{2 \pi}\left(|u|^{2}\left|\chi_{1}^{\prime}\right|^{2} / \varepsilon^{2}+|\nabla u|^{2}\left|\chi_{1}\right|^{2}\right) r d r d \theta\right. \\
& \left.+\int_{r_{0}}^{2 r_{0}} \int_{0}^{2 \pi}\left(|u|^{2}\left|\chi_{2}^{\prime}\right|^{2}+|\nabla u|^{2}\left|\chi_{2}\right|^{2}\right) r d r d \theta\right\} \tag{28}
\end{align*}
$$

and the same arguments as those used to control the term $J_{1}$ in (26) show that the two integrals above are $O(1)$. As for the last term in (27), we write

$$
\begin{aligned}
J_{4} & :=\int_{B_{2 \varepsilon} \backslash B_{\varepsilon}} a \nabla u \cdot v \nabla \chi_{\varepsilon} \\
& =\int_{B_{2 \varepsilon} \backslash B_{\varepsilon}} a \nabla u \cdot \bar{v} \nabla \chi_{\varepsilon}+\int_{B_{2 \varepsilon} \backslash B_{\varepsilon}} a \nabla u \cdot(v-\bar{v}) \nabla \chi_{\varepsilon},
\end{aligned}
$$

where $\bar{v}=\left|B_{2 \varepsilon}\right|^{-1} \int_{B_{2 \varepsilon}} v(x) d x$. We note that the first integral in the above right-hand side reduces to

$$
\bar{v} \int_{0}^{2 \pi} a(\theta) \phi(\theta) d \theta \int_{\varepsilon}^{2 \varepsilon} i \xi\left(\frac{r^{i \xi-1}-r^{-i \xi-1}}{2}\right) \frac{\chi_{1}^{\prime}(r / \varepsilon)}{\varepsilon} r d r=0 .
$$

Indeed, since $\phi$ is a solution to $\left(a(\theta) \phi^{\prime}(\theta)\right)^{\prime}-\xi^{2} a(\theta) \phi(\theta)=0$, with periodic boundary conditions, it satisfies

$$
\int_{0}^{2 \pi} a(\theta) \phi(\theta) d \theta=0
$$

It follows that

$$
\left|J_{4}\right| \leq\left(\int_{B_{2 \varepsilon} \backslash B_{\varepsilon}} a^{2}\left|\nabla u \cdot \nabla \chi_{\varepsilon}\right|^{2} d x\right)^{1 / 2}\left(\int_{B_{2 \varepsilon}}|v-\bar{v}|^{2}\right)^{1 / 2} .
$$

Using the following Poincaré inequality

$$
\int_{B_{2 \varepsilon}}|v-\bar{v}|^{2} \leq 4\left|B_{2 \varepsilon}\right|^{2} \int_{B_{2 \varepsilon}}|\nabla v|^{2}
$$

we obtain

$$
\begin{aligned}
\left|J_{4}\right| \leq & C \varepsilon\|v\|_{H^{1}(\Omega)}\left(\int_{0}^{2 \pi} a(\theta)^{2}|\phi(\theta)|^{2} d \theta\right)^{1 / 2} \\
& \left(\int_{\varepsilon}^{2 \varepsilon}\left|i \xi \frac{r^{i \xi-1}-r^{-i \xi-1}}{2}\right|^{2} \frac{\left[\chi^{\prime}(r / \varepsilon)\right]^{2}}{\varepsilon^{2}} r d r\right)^{1 / 2} \\
\leq & C\left(\int_{\varepsilon}^{2 \varepsilon} r^{-1} d r\right)^{1 / 2}\|v\|_{H^{1}(\Omega)} \\
\leq & C \sqrt{\ln (2)}\|v\|_{H^{1}(\Omega)}=O(1)\|v\|_{H^{1}(\Omega)}
\end{aligned}
$$

Altogether, 27,28 and the above estimate show that

$$
\forall v \in H_{0}^{1}(\Omega), \quad\left|\int_{\Omega} \nabla\left(\beta I-T_{D}\right) u_{\varepsilon} \cdot \nabla v\right| \leq O\left(s_{\varepsilon}\right)\|v\|_{H^{1}}
$$

which proves the claim since $s_{\varepsilon} \rightarrow 0$.
Finally, we show that $u_{\varepsilon} \rightarrow 0$ weakly in $H^{1}(\Omega)$. In fact, since this sequence is uniformly bounded in $H^{1}$, it suffices to show that $u_{\varepsilon} \rightarrow 0$ strongly in $L^{2}$, which follows from $(25)$, from the boundedness of $\chi_{1}$ and $\chi_{2}$ and from the fact that $s_{\varepsilon} \rightarrow 0$.

## 5 Characterization of the essential spectrum

In this section, we consider $\lambda \notin\left[\lambda_{-}, \lambda_{+}\right], \beta=1 / 2-\lambda$, and $k=(1-1 / \beta)$. The latter satisfies

$$
\begin{equation*}
k<k_{+}=\frac{-(2 \pi-\alpha)}{\alpha}<0 \quad \text { or } \quad k_{-}=\frac{-\alpha}{2 \pi-\alpha}<k<0 \tag{29}
\end{equation*}
$$

We show that $\beta \notin \sigma_{e s s}\left(T_{D}\right)$, so that according to Proposition $2, \lambda \notin \sigma_{e s s}\left(\mathcal{K}_{D}^{*}\right)$.

We proceed by contradiction: If $\beta \in \sigma_{e s s}\left(T_{D}\right)$, then there exists a singular Weyl sequence $u_{\varepsilon}$, that satisfies the conditions (23). In the next three sections, we show

Proposition 5. The sequence $u_{\varepsilon}$ converges to 0 strongly in $H^{1}(\Omega)$.
This contradicts the fact that $\left\|u_{\varepsilon}\right\|_{H^{1}}=1$. Consequently, in view of Theorem 2, this proves
Theorem 3. The essential spectrum of $\mathcal{K}_{D}^{*}$ is exactly

$$
\sigma_{e s s}\left(\mathcal{K}_{D}^{*}\right)=\left[\lambda^{-}, \lambda^{+}\right]
$$

### 5.1 Controling the energy of $u_{\varepsilon}$ away from the corner

Let $z_{\varepsilon}=\beta u_{\varepsilon}-T_{D} u_{\varepsilon} \in H_{0}^{1}(\Omega)$. Let $\rho<R_{0}$ and let $\chi_{\rho}$ denote a smooth, radial cut-off function, such that

$$
\chi_{\rho}(x)= \begin{cases}1 & \text { if }|x| \leq \rho / 2 \\ 0 & \text { if }|x| \geq \rho\end{cases}
$$

Let $v_{\varepsilon}=\left(1-\chi_{\rho}\right) u_{\varepsilon}$. We show that
Proposition 6. The sequence $v_{\varepsilon}$ converges strongly to 0 in $H^{1}$.
Proof: Assume that it is not the case. Then there exists $\delta>0$ and a subsequence (still labeled with $\varepsilon$ ) such that

$$
\begin{equation*}
\left\|v_{\varepsilon}\right\|_{H_{0}^{1}} \geq \delta \tag{30}
\end{equation*}
$$

We note that for any $v \in H_{0}^{1}(\Omega)$,

$$
\begin{align*}
\int_{\Omega} \nabla z_{\varepsilon} \cdot \nabla v & =\int_{\Omega} \nabla\left(\beta u_{\varepsilon}-T_{D} u_{\varepsilon}\right) \cdot \nabla v  \tag{31}\\
& =\beta \int_{\Omega} \nabla u_{\varepsilon} \cdot \nabla v+\int_{D} \nabla u_{\varepsilon} \cdot \nabla v \\
& =\int_{\Omega} a \nabla u_{\varepsilon} \cdot \nabla v \tag{32}
\end{align*}
$$

where $a(x)=\beta, x \in \Omega \backslash \bar{D}$, and $a(x)=\beta-1, x \in D$. Given $v \in H_{0}^{1}(\Omega)$, we compute

$$
\begin{aligned}
\int_{\Omega} a \nabla v_{\varepsilon} \cdot \nabla v & =\int_{\Omega} a \nabla\left[\left(1-\chi_{\rho}\right) u_{\varepsilon}\right] \cdot \nabla v \\
& =\int_{\Omega} a\left[\left(1-\chi_{\rho}\right) \nabla u_{\varepsilon}-u_{\varepsilon} \nabla \chi_{\rho}\right] \cdot \nabla v \\
& =\int_{\Omega} a \nabla u_{\varepsilon} \cdot\left[\nabla\left(\left(1-\chi_{\rho}\right) v\right)+v \nabla \chi_{\rho}\right]-a u_{\varepsilon} \nabla \chi_{\rho} \cdot \nabla v \\
& =\int_{\Omega} \nabla z_{\varepsilon} \cdot \nabla\left(\left(1-\chi_{\rho}\right) v\right)-u_{\varepsilon} \nabla \cdot\left(a v \nabla \chi_{\rho}\right)-a u_{\varepsilon} \nabla \chi_{\rho} \cdot \nabla v .
\end{aligned}
$$

Invoking the Cauchy-Schwarz and the Poincaré inequality, it follows that

$$
\begin{aligned}
\left|\int_{\Omega} \nabla\left(\left(\beta I-T_{D}\right) v_{\varepsilon}\right) \cdot \nabla v\right| & =\left|\int_{\Omega} a \nabla v_{\varepsilon} \cdot \nabla v\right| \\
& \leq C\left(\left\|u_{\varepsilon}\right\|_{L^{2}}+\left\|z_{\varepsilon}\right\|_{H_{0}^{1}}\right)\|v\|_{H_{0}^{1}}
\end{aligned}
$$

As $u_{\varepsilon} \rightarrow 0$ strongly in $L^{2}(\Omega)$ since it converges weakly to 0 in $H^{1}$, we conclude that

$$
\begin{equation*}
\left(\beta I-T_{D}\right) v_{\varepsilon}=\left(1-\chi_{\rho}\right) u_{\varepsilon} \rightarrow 0 \quad \text { strongly in } H_{0}^{1}(\Omega) . \tag{33}
\end{equation*}
$$

We note that since $v_{\varepsilon}$ has support in $\Omega \backslash B_{\rho / 2}$,

$$
T_{D} v_{\varepsilon}=T_{\tilde{D}} v_{\varepsilon},
$$

where $\tilde{D}$ denotes any smooth connected inclusion, such that $\left(D \backslash B_{\rho / 2}\right) \equiv$ ( $\tilde{D} \backslash B_{\rho / 2}$ ), and thus 33) also reads

$$
\left(\beta I-T_{\tilde{D}}\right) v_{\varepsilon}=\left(1-\chi_{\rho}\right) u_{\varepsilon} \rightarrow 0 \quad \text { strongly in } H_{0}^{1}(\Omega)
$$

It is easily seen that $v_{\varepsilon} \rightharpoonup 0$ weakly in $H_{0}^{1}(\Omega)$, and, upon rescaling in view of (30), we conclude from the above estimate that $v_{\varepsilon} /\left\|v_{\varepsilon}\right\|_{H_{0}^{1}}$ is a singular Weyl sequence for $T_{\tilde{D}}$. But $\tilde{D}$ is smooth, so that the associated NeumannPoincaré operator is compact and does not have essential spectrum, which contradicts this fact, and proves the Proposition.

### 5.2 Controling the energy of $u_{\varepsilon}$ near the corner

We now focus on $w_{\varepsilon}:=\chi_{\rho} u_{\varepsilon}$, which has compact support in $B_{\rho}$. In view of (31), it is easy to check that $w_{\varepsilon}$ satisfies

$$
\partial_{r r}^{2} w_{\varepsilon}+1 / r \partial_{r} w_{\varepsilon}+1 / r^{2} \partial_{\theta \theta}^{2} w_{\varepsilon}=\tilde{f}_{\varepsilon}
$$

in $\left(D \cap B_{\rho}\right)$ and in $\left((\Omega \backslash \bar{D}) \cap B_{\rho}\right)$. The right-hand side is defined as

$$
\tilde{f}_{\varepsilon}=\chi_{\rho} \Delta z_{\varepsilon}+b \nabla \chi_{\rho} \cdot \nabla u_{\varepsilon}+\nabla\left(b u_{\varepsilon}\right) \cdot \nabla \chi_{\rho}+b u_{\varepsilon} \Delta \chi_{\rho}
$$

and we note that it converges strongly to 0 in $H^{-1}(\Omega)$. Moreover, since the function $\chi_{\rho}$ is radial, $w_{\varepsilon}$ satisfies the following transmission conditions on the edges of the corner

$$
\left\{\begin{array}{c}
w_{\varepsilon}\left(r,\left.\frac{\alpha}{2}\right|_{-}\right)=w_{\varepsilon}\left(r,\left.\frac{\alpha}{2}\right|_{+}\right) \\
w_{\varepsilon}\left(r,-\left.\frac{\alpha}{2}\right|_{-}\right)=w_{\varepsilon}\left(r,-\left.\frac{\alpha}{2}\right|_{+}\right), \\
(\beta-1) \partial_{\theta} w_{\varepsilon}\left(r,\left.\frac{\alpha}{2}\right|_{-}\right)=\beta \partial_{\theta} w_{\varepsilon}\left(r,\left.\frac{\alpha}{2}\right|_{+}\right), \\
(\beta-1) \partial_{\theta} w_{\varepsilon}\left(r,-\left.\frac{\alpha}{2}\right|_{-}\right)=\beta \partial_{\theta} w_{\varepsilon}\left(r,-\left.\frac{\alpha}{2}\right|_{+}\right),
\end{array}\right.
$$

where the notations $\left.\right|_{-},\left.\right|_{+}$indicate taking the limit from left and right sides respectively.

We set

$$
\begin{equation*}
A=\frac{\alpha}{2 \pi-\alpha} \in(0,1) \tag{34}
\end{equation*}
$$

and consider the change of variables $(r, \theta) \in(0, \rho) \times(-\alpha / 2, \alpha / 2) \rightarrow(r, \pi-$ $\theta / A)$, which maps $D \cap B_{\rho}$ into $(\Omega \backslash \bar{D}) \cap B_{\rho}$. We define

$$
\left\{\begin{array}{l}
v_{\varepsilon}(r, \theta)=w_{\varepsilon}(r, \pi-\theta / A) \\
\tilde{g}_{\varepsilon}(r, \theta)=\tilde{f}_{\varepsilon}(r, \pi-\theta / A),
\end{array} \quad \text { for }(r, \theta) \in D \cap B_{\rho} .\right.
$$

It is easy to check that when $(f, g)=\left(\tilde{f}_{\varepsilon}, \tilde{g}_{\varepsilon}\right)$, the functions $(w, v)=$ $\left(\left.w_{\varepsilon}\right|_{D \cap B_{\rho}}, v_{\varepsilon}\right)$ satisfy the following system

$$
\left\{\begin{array}{l}
\partial_{r r}^{2} w+1 / r \partial_{r} w+1 / r^{2} \partial_{\theta \theta}^{2} w=f,  \tag{35}\\
\partial_{r r}^{2} v+1 / r \partial_{r} v+A^{2} / r^{2} \partial_{\theta \theta}^{2} v=g,
\end{array}\right.
$$

with the boundary conditions

$$
\begin{cases}v(\rho, \theta) & =w(\rho, \theta)=0  \tag{36}\\ v(r, \pm \alpha / 2) & =w(r, \pm \alpha / 2), \\ \partial_{\theta} v(r, \pm \alpha / 2) & =\frac{-k}{A} \partial_{\theta} w(r, \pm \alpha / 2) .\end{cases}
$$

In other words, $w_{\varepsilon}$ and $u_{\varepsilon}$ both satisfy an elliptic equation and take nearly the same Cauchy data on the edges of the corner.
To study the above system, we introduce the (closed) subspace $V_{1} \subset H^{1}(D \cap$ $\left.B_{\rho}\right) \times H^{1}\left(D \cap B_{\rho}\right)$ of functions $(w, v)$ that satisfy

$$
\left\{\begin{array}{lll}
v(\rho, \theta) & =w(\rho, \theta)=0 & |\theta|<\alpha / 2 \\
v(r, \pm \alpha / 2) & =w(r, \pm \alpha / 2) & 0<r<\rho .
\end{array}\right.
$$

Theorem 4. The system 35 366) has a unique solution $(w, v) \in V_{1}$. Moreover, there exists a constant $C>0$, such that

$$
\|\nabla w\|_{L^{2}\left(D \cap B_{\rho}\right)}+\|\nabla v\|_{L^{2}\left(D \cap B_{\rho}\right)} \leq C\left(\|f\|_{H^{-1}\left(D \cap B_{\rho}\right)}+\|g\|_{H^{-1}\left(D \cap B_{\rho}\right)}\right) .
$$

Proof: On $V_{1}$ we consider the norm

$$
\begin{equation*}
\|(w, v)\|:=\left(\int_{D \cap B_{\rho}}|\nabla w|^{2}+|\nabla v|^{2}\right)^{1 / 2} . \tag{37}
\end{equation*}
$$

We multiply the equations by by two functions $\phi, \psi \in H^{1}\left(D \cap B_{\rho}\right)$ that vanish on $D \cap \partial B_{\rho}$, and integrate to obtain

$$
\begin{aligned}
& \int_{D \cap B_{\rho}} f \phi+g \psi \\
& =\int_{0}^{\rho} \int_{-\alpha / 2}^{\alpha / 2}\binom{\partial_{r} w}{\frac{1}{r} \partial_{\theta} w} \cdot\binom{\partial_{r} \phi}{\frac{1}{r} \partial_{\theta} \phi}+\binom{\partial_{r} v}{\frac{A^{2}}{r} \partial_{\theta} v} \cdot\binom{\partial_{r} \psi}{\frac{1}{r} \partial_{\theta} \psi} r d r d \theta \\
& \quad-\int_{\theta= \pm \alpha / 2} \frac{1}{r} \partial_{\theta} w \phi+\frac{A^{2}}{r} \partial_{\theta} v \psi
\end{aligned}
$$

We note that that the last integral can be rewritten as

$$
\int_{\theta= \pm \alpha / 2} \frac{1}{r} \partial_{\theta} w(\phi-A k \psi)+\frac{A^{2}}{r}\left[\partial_{\theta} v+\frac{k}{A} \partial_{\theta} w\right] \psi .
$$

To satisfy the natural boundary condition in (36), we are thus led to introduce the subspace $V_{2} \subset H^{1}\left(D \cap B_{\rho}\right) \times H^{1}\left(D \cap B_{\rho}\right)$ of functions $(\phi, \psi)$ that satisfy

$$
\begin{cases}\phi(\rho, \theta)=\psi(\rho, \theta)=0, & |\theta|<\alpha / 2 \\ \phi\left(r, \pm \frac{\alpha}{2}\right)-A k \psi\left(r, \pm \frac{\alpha}{2}\right)=0, & 0<r<\rho\end{cases}
$$

which we also equip with the norm (37). We also introduce the following bilinear form $B$ on $V_{1} \times V_{2}$ by

$$
\begin{aligned}
B\left(\binom{w}{v},\binom{\phi}{\psi}\right)= & \int_{0}^{\rho} \int_{-\alpha / 2}^{\alpha / 2}\binom{\partial_{r} w}{\frac{1}{r} \partial_{\theta} w} \cdot\binom{\partial_{r} \phi}{\frac{1}{r} \partial_{\theta} \phi} \\
& +\binom{\partial_{r} v}{\frac{A^{2}}{r} \partial_{\theta} v} \cdot\binom{\partial_{r} \psi}{\frac{1}{r} \partial_{\theta} \psi} r d r d \theta
\end{aligned}
$$

Thus, solving (35 36) amounts to solving the variational problem : find $W=(w, v) \in V_{1}$ such that

$$
\forall \Phi=(\phi, \psi) \in V_{2}, \quad B(W, \Phi)=\int_{D \cap B_{\rho}} f \phi+g \psi
$$

It is easily checked that the above right-hand side defines a continuous linear form on $V_{2}$, and that

$$
\forall(W, \Phi) \in V_{1} \times V_{2}, \quad|B(W, \Phi)| \leq\|W\|\|\Phi\| .
$$

Therefore, the theorem will be proved upon showing that $B$ satisfies the infsup condition (for instance the version in [7]), i.e., that there exists $\delta>0$ such that

$$
\begin{equation*}
\inf _{W \in V_{1},\|W\|=1}\left(\sup _{\Phi \in V_{2},\|\Phi\|=1} B(W, \Phi)\right) \geq \delta \tag{38}
\end{equation*}
$$

Let $W=(w, v) \in V^{1}$ and $p, q, d \in \mathbf{R}$. We set

$$
\phi=(A k p+d) w+(A k q-d) v, \quad \psi=p w+q v
$$

so that $\phi-A k \psi=d(w-v)=0$, and thus $(\phi, \psi) \in V_{2}$ is an admissible test function. The integrand in the expression of $B(W, \phi)$ takes the form

$$
\begin{aligned}
e:= & \partial_{r} u \partial_{r}[(A k p+d) w+(A k q-d) v]+\partial_{r} v \partial_{r}(p w+q v) \\
& +r^{-2} \partial_{\theta} u \partial_{\theta}[(A k p+d) w+(A k q-d) v]+\frac{A^{2}}{r^{2}} \partial_{\theta} v \partial_{\theta}(p w+q v) \\
= & (A k p+d) \xi_{1}^{2}+(A k q-d+p) \xi_{1} \xi_{3}+q \xi_{3}^{2} \\
& +(A k p+d) \xi_{2}^{2}+\left(A k q-d+A^{2} p\right) \xi_{2} \xi_{4}+A^{2} q \xi_{4}^{2}
\end{aligned}
$$

where $\xi_{1}=\partial_{r} \phi, \xi_{2}=r^{-1} \partial_{\theta} \phi, \xi_{3}=\partial_{r} \psi, \xi_{4}=r^{-1} \partial_{\theta} \psi$. Fixing $q=1$, it follows that $e$ defines a positive definite quadratic form (pointwise) provided that the polynomials

$$
\begin{aligned}
& P_{1}(\xi)=(A k p+d)+(A k-d+p) \xi+\xi^{2} \\
& P_{2}(\xi)=(A k p+d)+\left(A k-d+A^{2} p\right) \xi+A^{2} \xi^{2}
\end{aligned}
$$

are strictly positive, in other words, provided that

$$
\begin{cases}(A k-d+p)^{2}-4(A k p+d) & <0  \tag{39}\\ \left(A k-d+A^{2} p\right)^{2}-4 A^{2}(A k p+d) & <0\end{cases}
$$

We regard these expressions as polynomials in $p$, the roots of which are respectively

$$
\begin{aligned}
f_{ \pm}(d) & =(d+A k) \pm 2 \sqrt{d(1+A k)} \\
g_{ \pm}(d) & =\frac{1}{A^{2}}[(d+A k) \pm 2 A \sqrt{d(1+k / A)}]
\end{aligned}
$$

We remark that the roots are real if and only if

$$
\begin{cases}k<-1 / A & \text { if } \lambda<0 \\ -A<k<0 & \text { if } \lambda>0\end{cases}
$$

i.e., recalling (34, 29), if and only if $\lambda \notin\left[\lambda_{-}, \lambda_{+}\right]$, which is our hypothesis.

It only remains to show that we can indeed find parameters $p, d$ for which (39) is satisfied, i.e. that we can find $d$ such that

$$
\begin{equation*}
\left(f_{-}(d), f_{+}(d)\right) \cap\left(g_{-}(d), g_{+}(d)\right) \neq \emptyset \tag{40}
\end{equation*}
$$

(and then pick $p$ in the intersection).
To this end, assume first that $-A<k<0$, so that $d_{+}:=-A k>0$. We note that

$$
\frac{f_{+}\left(d_{+}\right)+f_{-}\left(d_{+}\right)}{2}=0,
$$

and that

$$
\begin{aligned}
& g_{+}\left(d_{+}\right)=\frac{2}{A^{2}} \sqrt{A k(k / A-1)}>0, \\
& g_{-}\left(d_{+}\right)=\frac{-2}{A^{2}} \sqrt{A k(k / A-1)}<0,
\end{aligned}
$$

which yields 40).
If $k<-1 / A$, one can see that $d_{-}=A^{2}+k A<0$ and that

$$
\frac{f_{+}\left(d_{-}\right)+f_{-}\left(d_{-}\right)}{2}=2 A k+A^{2}<-1=g_{p}\left(d_{-}\right) .
$$

On the other hand, since $0<A<1$ and $k<-1 / A$, we have

$$
\frac{-2}{A} \sqrt{d(1+k / A)}<-2 \sqrt{d(1+A k)},
$$

so that for any $d<0, g_{-}(d)<f_{-}(d)$, and in particular $g_{-}\left(d_{-}\right)<\frac{f_{+}\left(d_{-}\right)+f_{-}\left(d_{-}\right)}{2}$. It follows that 40 also holds in this case.

### 5.3 Proof of Proposition 5

We come back to the singular Weyl sequence $u_{\varepsilon}$, which we split as $u_{\varepsilon}=$ $\left(1-\chi_{\rho}\right) u_{\varepsilon}+\chi_{\rho} u_{\varepsilon}$. Proposition 6 shows that $\left(1-\chi_{\rho}\right) u_{\varepsilon}$ converges strongly to 0 . On the other hand, Theorem applied to $\chi_{\rho} u_{\varepsilon}$ shows that

$$
\begin{aligned}
\left\|\nabla\left(\chi_{\rho} u_{\varepsilon}\right)\right\|_{L^{2}\left(B_{\rho}\right)} & \leq C\left(\left\|\tilde{f}_{\varepsilon}\right\|_{H^{-1}\left(D \cap B_{\rho}\right)}+\left\|\tilde{g}_{\varepsilon}\right\|_{H^{-1}\left(D \cap B_{\rho}\right)}\right) \\
& \leq C\left(\left\|z_{\varepsilon}\right\|_{H^{1}(\Omega)}+\left\|u_{\varepsilon}\right\|_{L^{2}(\Omega)}\right) \rightarrow 0 .
\end{aligned}
$$

It thus follows that $u_{\varepsilon}$ converges strongly to 0 in $H^{1}(\Omega)$, which contradicts the assumption that $\left\|u_{\varepsilon}\right\|_{H^{1}(\Omega)}=1$, so that $\beta \notin \sigma_{\text {ess }}\left(T_{D}\right)$.

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