Lefschetz sequences and chaotic dynamics

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Motivation: Topological methods for detecting chaotic dynamics.

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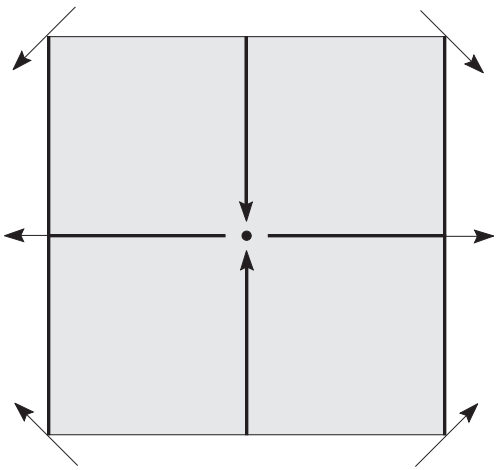

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The *Poincaré map* of the equation

$$(*) \qquad \dot{x} = f(t, x)$$

is the function

$$P\colon x_0 \mapsto P(x_0) := \varphi_{(0,T)}(x_0).$$

$\varphi_{(0,T)}(x_0)$ is the value of the solution of the problem

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Remark

$\{k\text{-periodic points of } P\} =$
$\{\text{Initial points of } kT\text{-periodic solutions of } (*)\}$

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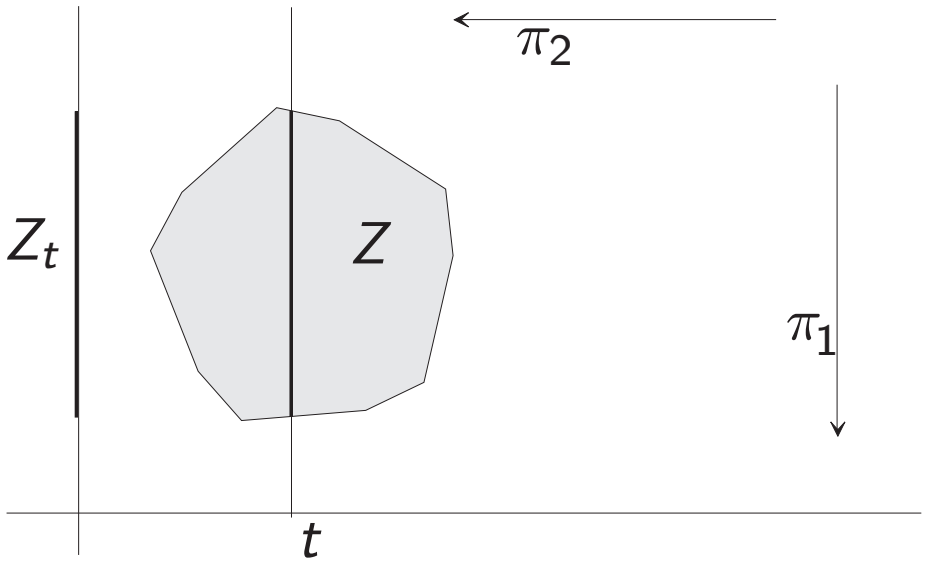

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if

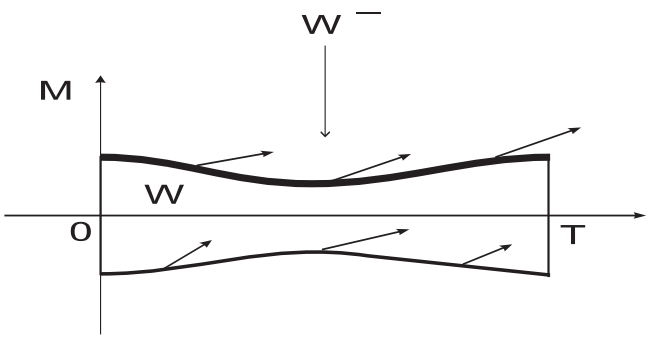
(1) W is a block for the vector-field $\begin{bmatrix} 1 \\ f \end{bmatrix}$,

(2) there is a compact set W^{--} contained in W such that

$$W^{-} = W^{--} \cup (\{T\} \times W_T),$$

(3) there exists a homeomorphism $h\colon [0, T] \times W_0 \to W$ such that $\pi_1 \circ h = \pi_1$ and $h([0, T] \times W_0^{--}) = W^{--}$,

(4) $W_0 = W_T$, $W_0^{--} = W_T^{--}$ are compact ENR's.


w
M
w
o
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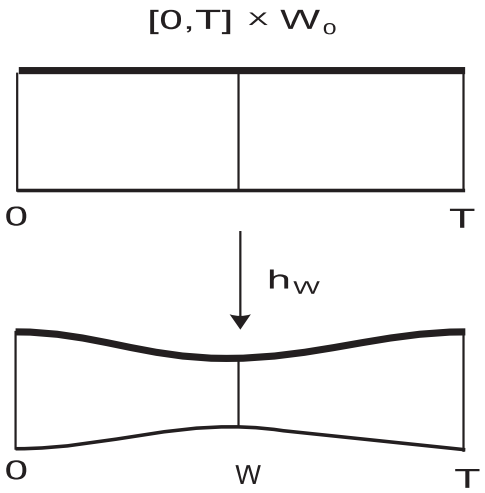
[0,T] × W₀
o
T
h_w
o
w
T

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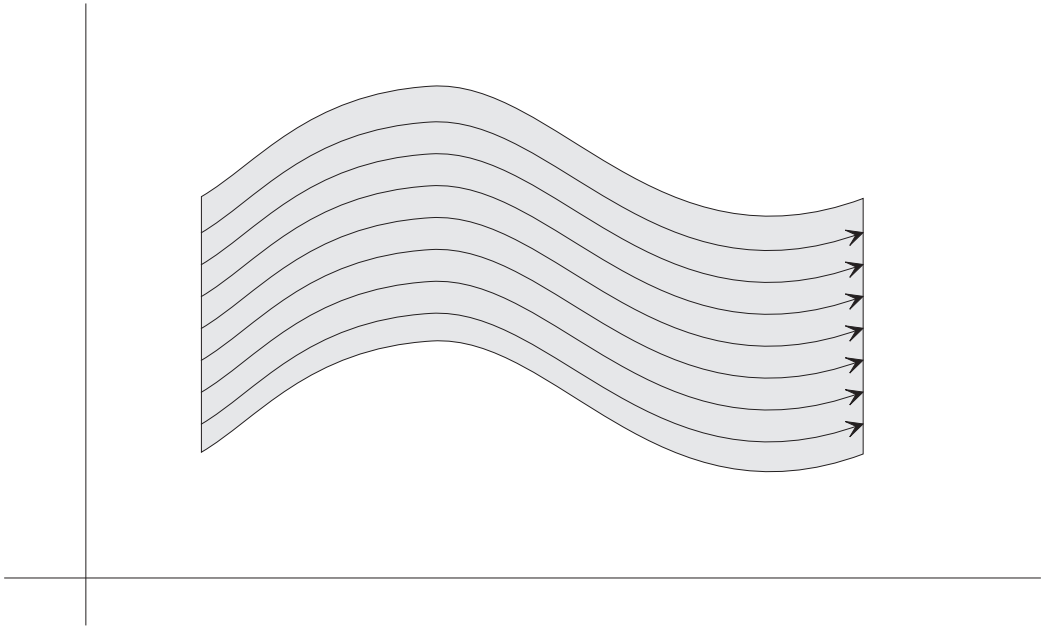

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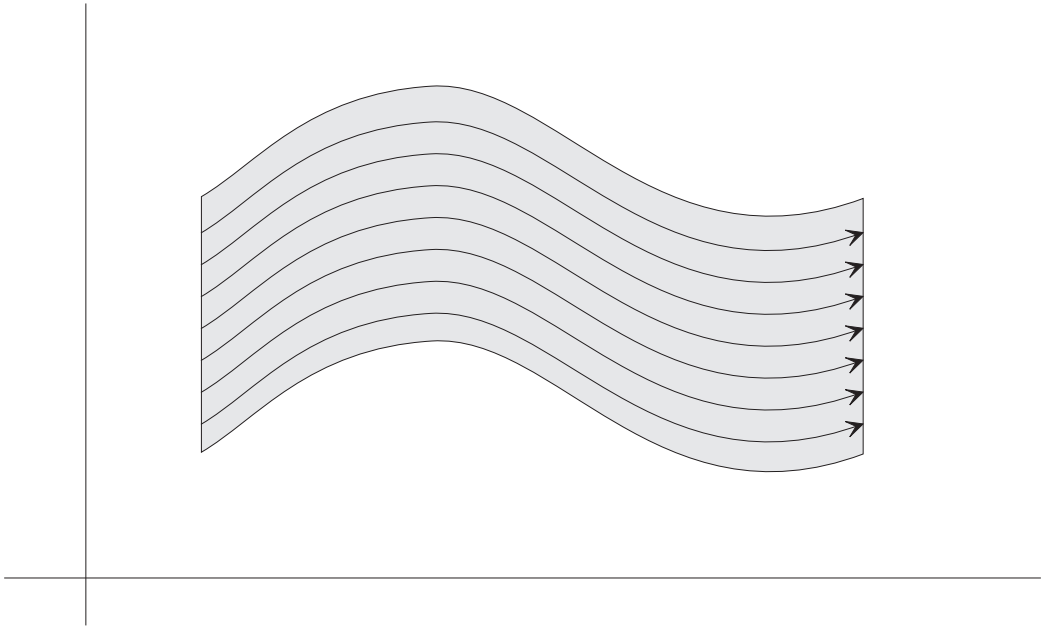

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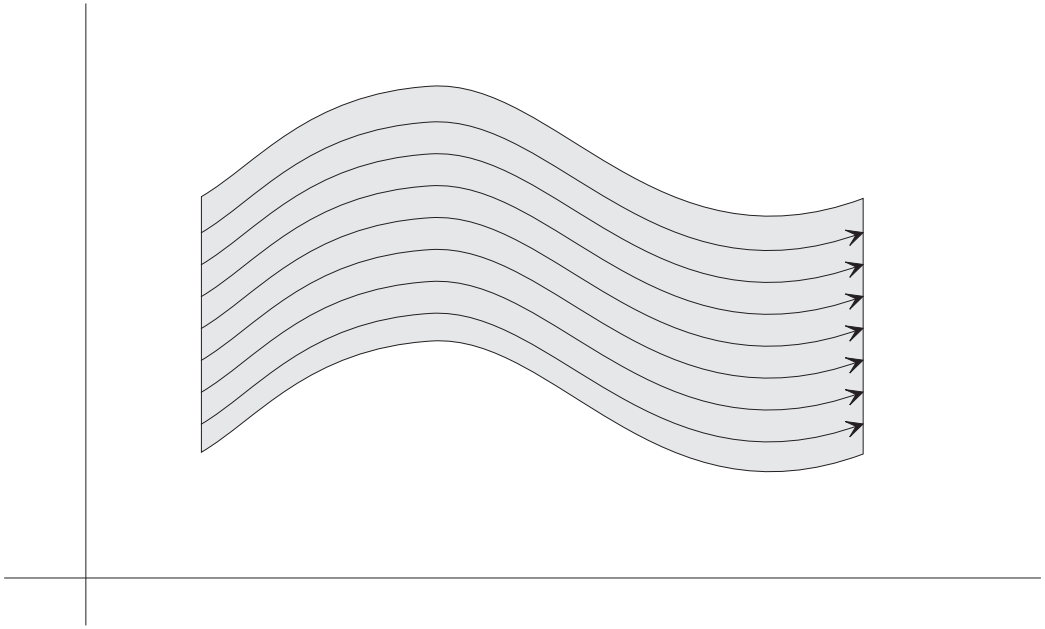

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$$\mu_W := H(m)\colon H(W_0, W_0^{--}) \to H(W_0, W_0^{--})$$

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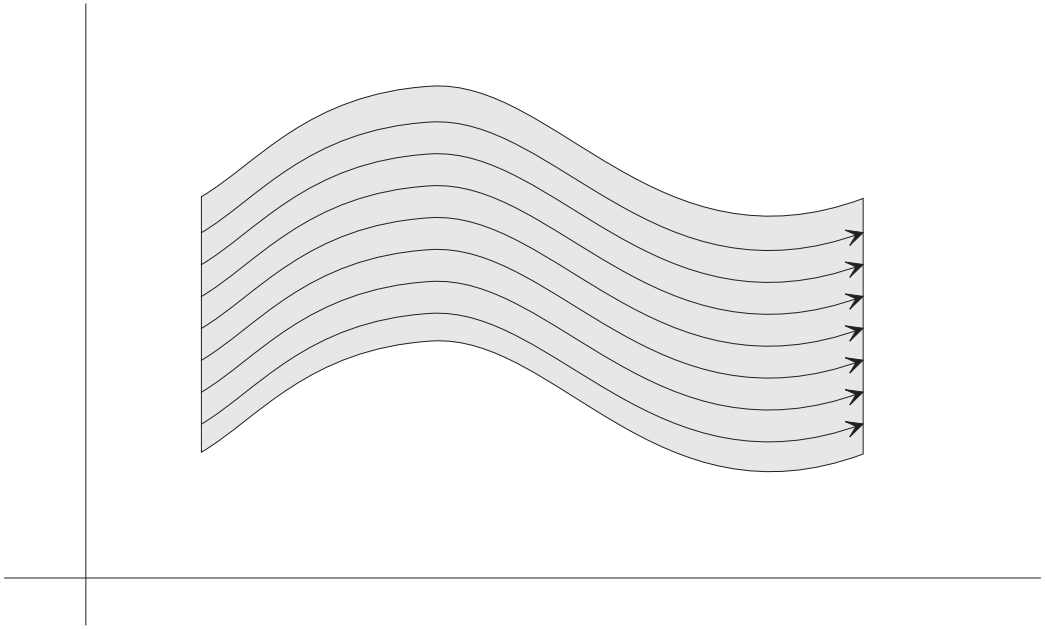

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$$L(\mu_W) := \sum_{n=0}^{\infty} (-1)^n \operatorname{trace} H_n(m)$$

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Definition

The equation

$$\dot{x} = f(t,x)$$

is Σ_2-*chaotic* if there exists a compact set $I \subset \mathbb{M}$, invariant with respect to the Poincaré map P and a function $g\colon I \to \Sigma_2$ such that

(1) g is continuous and surjective,

(2) $\sigma \circ g = g \circ P$,

(3) for every k-periodic sequence $s \in \Sigma_2$ the set $g^{-1}(s)$ contains at least one k-periodic point of P.

Let W and U be segments over $[0, T]$ for the equation

$$(*) \qquad \dot{x} = f(t, x),$$

where f is smooth and T-periodic with respect to t and

(1) $(W_0, W_0^{--}) = (U_0, U_0^{--})$,

(2) $U \subset W$,

(3) $\mu_U = \mathrm{id}_{H(W_0, W_0^{--})}$.

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$$g(x)_n = \begin{cases} 1, & \text{if } P^n(x_0) \text{ leaves } U \text{ in time less then } T, \\ 0, & \text{otherwise.} \end{cases}$$

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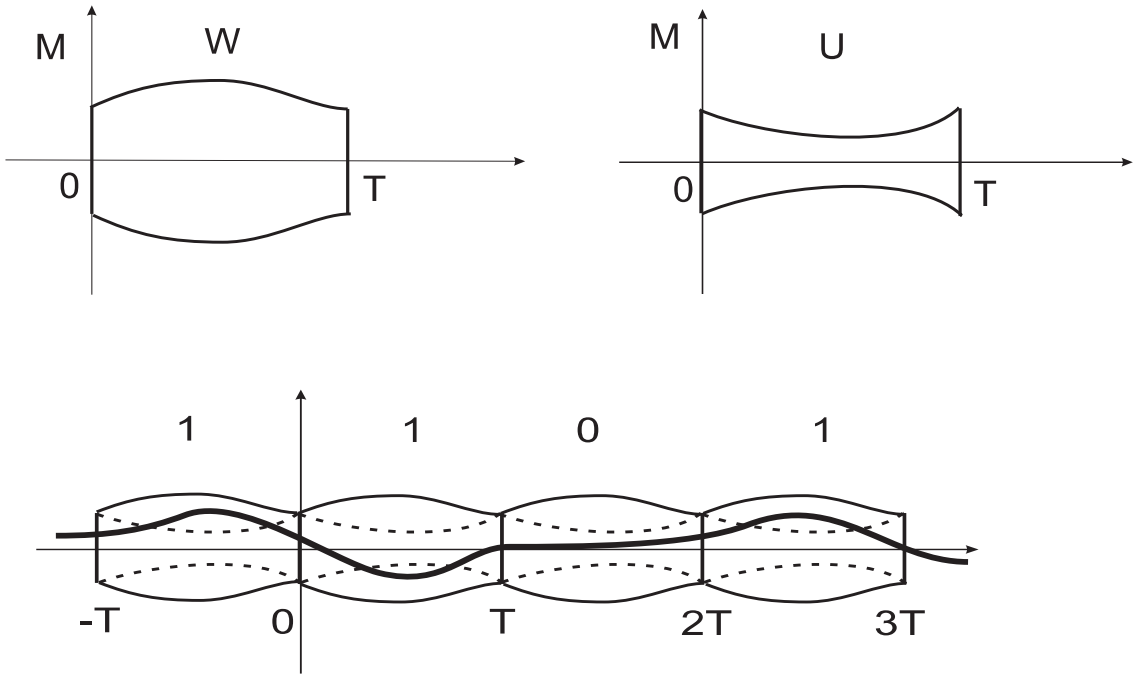
One can prove that $g : I \to \Sigma_2$ is continuous and

$$\sigma \circ g = g \circ P.$$


M
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Theorem (Srzednicki, KW)

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Corollary

If $L_1 \neq L_0$ then $g : I \to \Sigma_2$ is surjective. Moreover, if $p > |L_1 - L_0|$ is prime, then $L_p^* \neq 0$.

If L_k is constant, then $L_k^* = 0$ for $k \geq 1$.

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Example

For 2-periodic L_k we have

$$L_k^* = \begin{cases} L_0, & \text{if } k = 0; \\ (-2)^{k-1}(L_1 - L_0), & k \geq 1, \end{cases}$$

Theorem (Pieniążek,KW)

Assume that L_n is m-periodic and

$$L_1 = \ldots = L_{m-1}, \quad 0 \neq L_m = L_0 \neq L_1.$$

If m is even then $L_k^* \neq 0$ for $k \geq 0$.

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If m is even then $L_k^* \neq 0$ for $k \geq 0$.

If m is odd then $L_k^* = 0$ iff k is odd multiplicity of m.

Theorem (Pieniążek, Srzednicki,KW)

For $n \geq 2$ and $\phi > 0$ small enough the equation

$$\dot{z} = (1 + e^{i\phi t}|z|^2)\overline{z}^n$$

is Σ_2-chaotic for odd n. If n is even, then g is surjective and $F(c_k^m) \neq \emptyset$ whenever k is not an odd multiplicity of $m = n + 1$.

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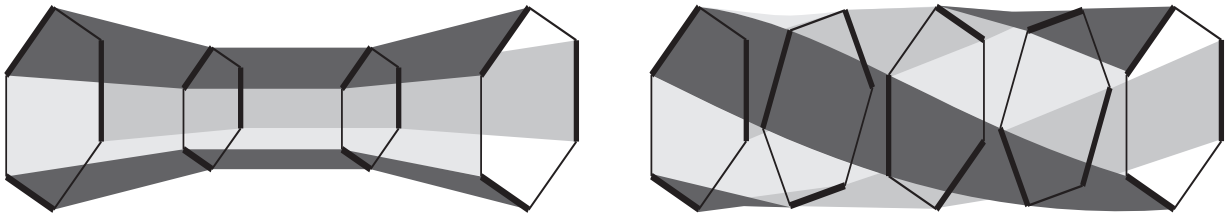

$(n = 2)$

Möbius function $\mu : \mathbb{N} \to \mathbb{Z}$ is given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k,\ p_i \text{ different primes}, \\ 0 & \text{otherwise.} \end{cases}$$

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A sequence a_n of integers satisfies the Dold's relations if

$$\forall n \in \mathbb{N} \quad \sum_{d|n} \mu(n/d) a_d \cong 0 \mod n.$$

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In particular,

$$a_p \cong a_1 \mod p,$$

for prime p.

Assume that p is odd prime and L_k is p-periodic and non-constant.

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Then

$$L_1 = \ldots = L_{p-1}, \quad L_p = L_0 \neq L_1.$$

It follows that $L_k^* = 0$ if and only if k is odd multiplicity of p.

Lemma

Assume that a_n is m-periodic sequence of integers such that for each prime p

$$a_p \cong a_1 \bmod p. \tag{1}$$

If $(r, m) = 1$, then $a_r = a_1$

Lemma

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Proof: Since $(r, m) = 1$, so by Dirichlet's theorem there are $n_k \to \infty$ such that

$$p_k = r + n_k m$$

is a prime number for each $k \in N$. In particular, there is k such that

$$p_k > |a_r - a_1|.$$

Then, by

$$a_r = a_{p_k} \cong a_1 \bmod p_k$$

we get that $p_k | a_r - a_1$, hence $a_r = a_1$.

Theorem (Marzantowicz, KW)

(a) *Sequences L_n and L_n^* satisfy Dold's relations,*

(b) *If L_n is bounded but not the constant sequence (so it is periodic), then L_n^* is unbounded.*

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(a) *Sequences L_n and L_n^* satisfy Dold's relations,*

(b) *If L_n is bounded but not the constant sequence (so it is periodic), then L_n^* is unbounded. Moreover there exists $\rho > 1$ such that*

$$\lim_{k\to\infty} \frac{L^*_{2lk}}{\rho^{2lk}} = a \neq 0.$$

For $n \geq |L_1 - L_0|$ we put

$$S(n) = \{p \in \mathbb{N} : |L_1 - L_0| < p < n, p \not| n,\ p \text{ prime}\}.$$

and

$$\nu(n) = \sum_{p \in S(n)} \binom{n}{p}.$$

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Theorem (Gierzkiewicz, KW)

If $n \geq |L_1 - L_0|$, then P has at least $\nu(n)$ periodic points with minimal period n.

Theorem

Let $k \geq 1$ be fixed. If the sequence $\{L^*_{mk}\}_{m\geq 1}$ is unbounded, then for each n-periodic sequence c^n_k the set $g^{-1}(c^n_k)$ contains infinitely many periodic points of P.

Corollary

Assume that L_n is a bounded but not the constant sequence. Then, $g^{-1}(1^\infty)$ contains infinitely many periodic points of P.