Homeomorphism groups of commutator width one

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Definition

A group is *uniformly perfect* if every element is written as a product of a bounded number of commutators. The least number of such bound is called the *commutator width*.

Theorem

Any element of $\mathbf{Homeo}(S^n)_0$ can be written as one commutator, where $\mathbf{Homeo}(S^n)_0$ is the identity component of the group of homeomorphisms of the n-dimensional sphere S^n.

Theorem

Any element of $\mathbf{Homeo}(\mu^n)$ can be written as one commutator, where $\mathbf{Homeo}(\mu^n)$ is the group of homeomorphisms of the n-dimensional Menger compact space μ^n.

$\mathbf{Homeo}(S^1)_0$

David Eisenbud, Ulrich Hirsch and Walter Neumann showed in

[*Transverse foliations of Seifert bundles and self homeomorphism of the circle*, Comment. Math. Helvetici 56 (1981) 638–660]

that the commutator width of $\mathbf{Homeo}(S^1)_0$ is one.

$\mathbf{Homeo}_c(\mathbf{R}^n)$

In 1958, Anderson showed that in the group $\mathbf{Homeo}_c(\mathbf{R}^n)$ of homeomorphisms of the n-dimensional Euclidean space $\mathbf{R}^n$ with compact support, any element can be written as one commutator.

Then it has been known that the commutator width of $\mathbf{Homeo}(S^n)_0$ is at most 2.

The construction by Anderson

It is worth recalling the construction by Anderson.

For given $f \in \mathrm{Homeo}_c(\mathbf{R}^n)$, we find a bounded ball U such that the support $\mathrm{supp}(f) \subset U$.

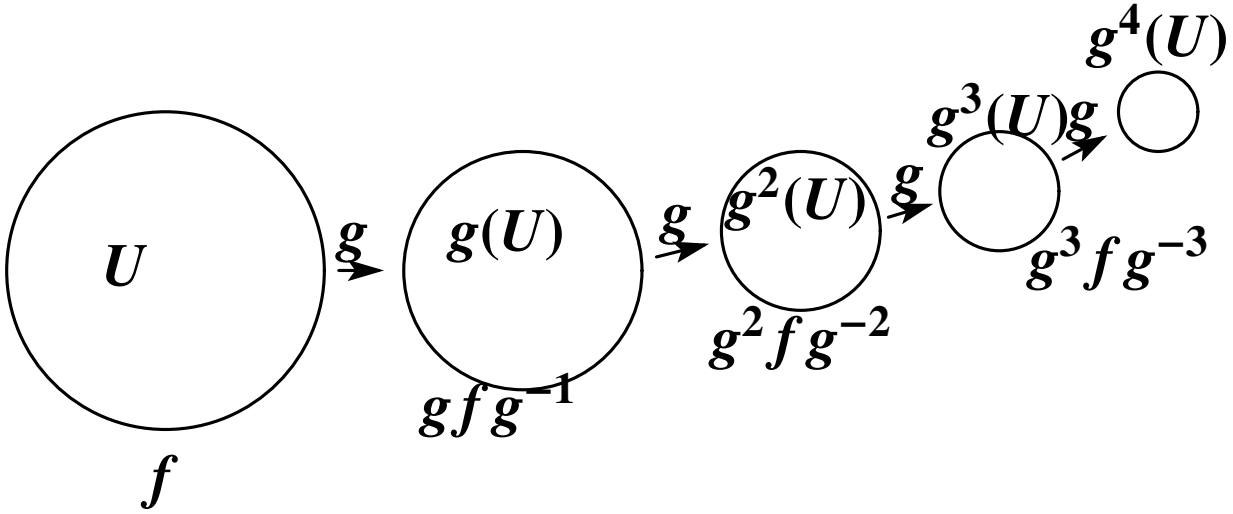
Then we can find an element $g \in \mathrm{Homeo}_c(\mathbf{R}^n)$ such that

(1) $g^n(U)$ $(n \in \mathbf{Z})$ are disjoint and
(2) $\lim_{n\to\infty} \mathrm{diam}(g^n(U)) = 0$.

Put $F = \prod_{n=0}^{\infty} g^n f g^{-n}$, then we have $gFg^{-1} = f^{-1}F$.

Thus $f = FgF^{-1}g^{-1}$.

The construction by Anderson



Meaning???

What is the meaning that the commutator width is one?

In the case of $\mathbf{Homeo}_c(\mathbf{R}^n)$, we see that for any element f, there exist g such that g and fg are conjugate.

That is, g is dynamically so strong that fg and g have the same dynamics, and hence they are conjugate.

Strong dynamics

In the case of $\mathbf{Homeo_0}(S^n)$ or $\mathbf{Homeo}(\mu^n)$, we have the candidate which has the strong dynamics.

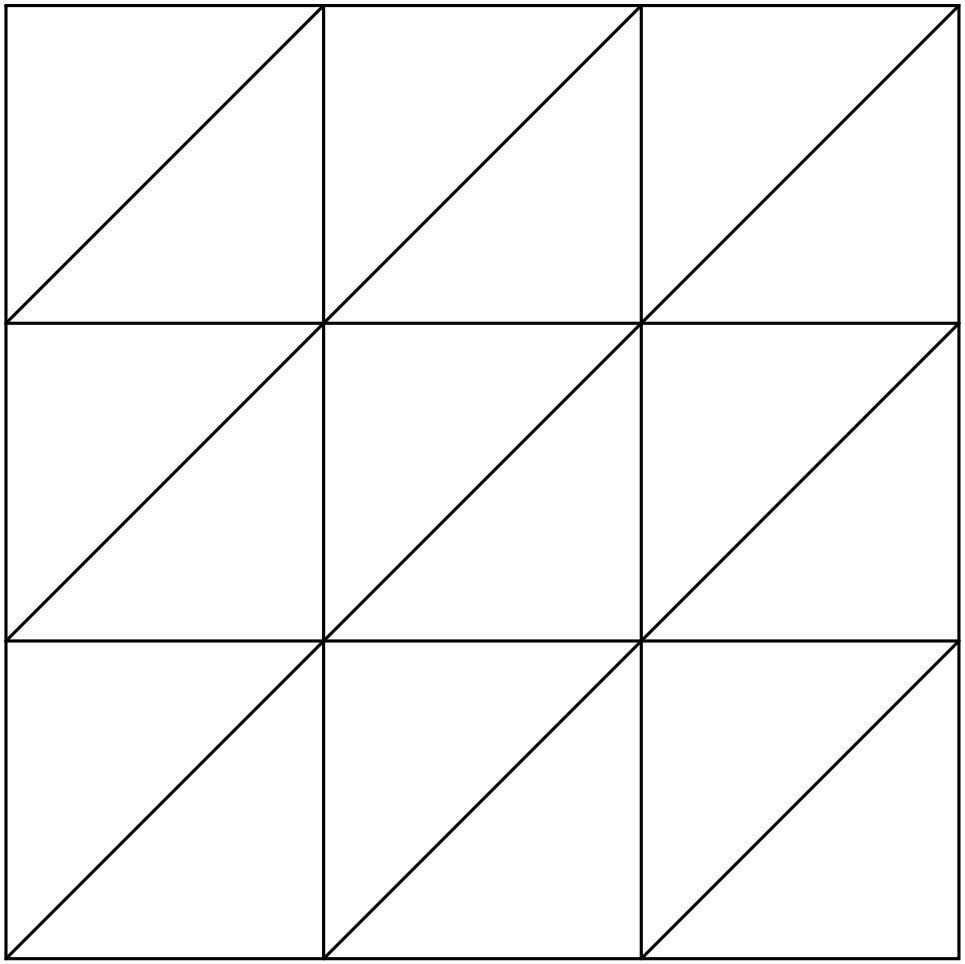
The candidate is the topologically hyperbolic homeomorphism.

Definition

A topologically hyperbolic homeomorphism is a homeomorphism h with one source s_+ and one sink s_- such that $\lim_{n\to+\infty} h^n(x) = s_-$ and $\lim_{n\to-\infty} h^n(x) = s_+$ for $x \notin \{s_-, s_+\}$.

Orientation preserving topologically hyperbolic homeomorphisms with nice fundamental domains outside the fixed points are conjugate.

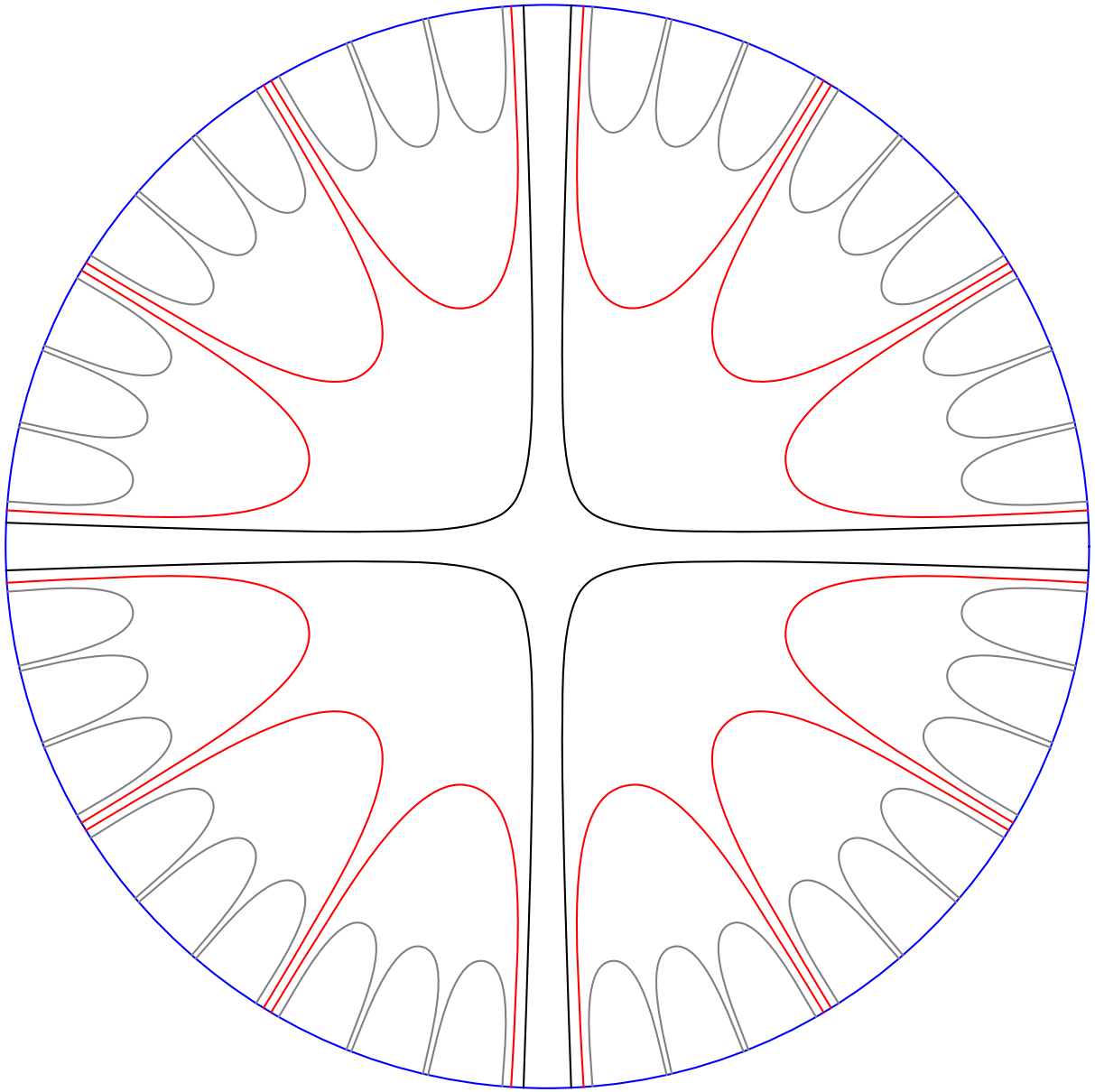
$\mathrm{Diffeo}(S^1)_0$



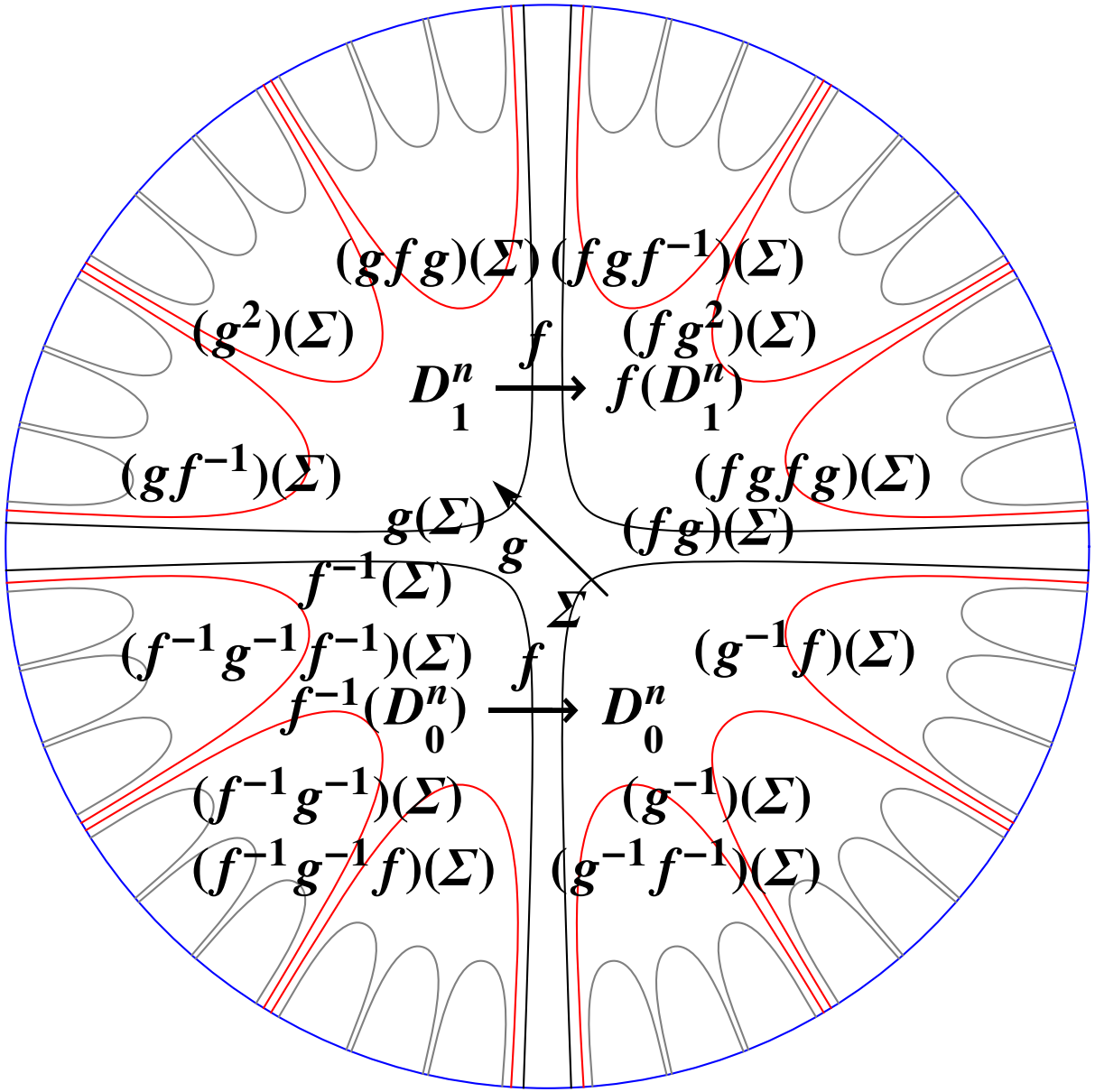
Hence for a given homeomorphism f, we will construct a topologically hyperbolic homeomorphism g which is so strong that fg is topologically hyperbolic. See Figure.

Take four disjoint disks $f^{-1}(D^n_0)$, D^n_0, D^n_1, $f(D^n_1)$ and define g such that $g(D^n_0) = S^n \setminus \mathrm{int}(D^n_1)$ and $g(S^n \setminus \mathrm{int}(D^n_0)) = D^n_1$, and observe how are the actions of g and fg.

$\mathrm{Diffeo}(S^n)_0$



$\mathrm{Diffeo}(S^n)_0$



For the proof

In the construction, we need the following deep theorems.

Generalized Schoenflies Theorem (Brown)

Let Σ be a locally flat $(n-1)$-dimensional sphere in the n-dimensional sphere S^n. Then the closures of the complementary domains of Σ are homeomorphic to the n-dimensional disk D^n.

Annulus conjecture (Kirby, Quinn)

Let Σ_0 and Σ_1 be disjoint locally flat $(n-1)$-dimensional spheres in the n-dimensional sphere S^n. Then the closure of the region between them is homeomorphic to $S^{n-1} \times [0,1]$.

For the proof

Lemma

For any compact set K in the interior $\mathbf{int}(D^n)$ of the standard disk D^k and any positive real number ε, there is a homeomorphism $\varphi_{K,\varepsilon} : D^n \longrightarrow D^n$ which is the identity on ∂D^n such that $\mathbf{diam}(\varphi_{K,\varepsilon}(K)) \leq \varepsilon$.

Menger compact spaces

For the n-dimensional Menger compact space μ^n, we use the following propositions in the place of the above theorems and lemma. They are shown by using Bestvina's Z-set unknotting theorem.

Proposition

Let A be a closed set in the compact n-dimensional Menger space μ^n such that

(1) A is homeomorphic to the compact $(n-1)$-dimensional Menger space μ^{n-1},

(2) $\mu^n \setminus A = U_1 \cup U_2$, $U_1 \neq \emptyset$, $U_2 \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$, and

(3) $U_1 \cup A$ and $U_2 \cup A$ are n-dimensional Menger manifolds and A is a Z-set in $U_1 \cup A$ and in $U_2 \cup A$.

Then $U_1 \cup A$ and $U_2 \cup A$ are homeomorphic to μ^n.

Prpopsition

Let A_1 and A_2 be a closed set in the compact n-dimensional Menger space μ^n such that

(1) A_1 and A_2 are homeomorphic to the disjoint union of two compact $(n-1)$-dimensional Menger space μ^{n-1},

(2) $\mu^n \setminus A_i = U_{i1} \cup U_{i2} \cup U_{i3}$ (disjoint union of nonempty open sets; $i = 1, 2$) and

(3) $\overline{U_{i1}}$, $\overline{U_{i2}}$ and $\overline{U_{i3}}$ are n-dimensional Menger manifolds and $A_i \subset \overline{U_{i2}}$ is a Z-set as well as $A_i \cap \overline{U_{i1}} \subset \overline{U_{i1}}$ and $A_i \cap \overline{U_{i3}} \subset \overline{U_{i3}}$.

Then any homeomorphism $A_1 \longrightarrow A_2$ extends to a homeomorphism $h : \mu^n \longrightarrow \mu^n$ such that $h(U_{1j}) = U_{2j}$ after changing the indices $_{i1}$ and $_{i3}$ if necessary.

Lemma

Under the assumption of the former Proposition for any compact set K in U_1 and any positive real number ε, there is a homeomorphism $\varphi_{K,\varepsilon}$ of μ^n such that $\varphi_{K,\varepsilon}|(U_2 \cup A) = \mathrm{id}_{U_2 \cup A}$, $\mathrm{diam}(\varphi_{K,\varepsilon}(K)) \leq \varepsilon$.