Homotopy Techniques for Tensor Decomposition and Perfect Identifiability

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Tensors, Rank, and Identifiability

- A tensor of format \((n_1, n_2, \ldots, n_d)\) is a hypermatrix \(\mathcal{T} = (\mathcal{T}_{i_1,i_2,\ldots,i_d})\) (assume entries in \(\mathbb{C}\)), with \(1 \leq i_j \leq n_j\) for all \(j\).
- Since \(\mathcal{T}\) has \(\prod_{j=1}^{d} n_j\) entries, it represents a huge set of data.
- Basic question: Find a sparse representation of \(\mathcal{T}\).
- A rank-one tensor (a point on a Segre variety) is a tensor \(\mathcal{T}\) such that \(\mathcal{T}_{i_1,\ldots,i_d} = v_{1,j_1} \cdot v_{2,j_2} \cdots v_{d,j_d}\) for some vectors \(\vec{v}_j\) of lengths \(n_j\).
- Rank-one tensors only have essentially \(\sum_{j=1}^{d} n_j\) pieces of information.
- A rank-\(r\) tensor (a general point on the \(r\)-th secant variety of the Segre variety) is the sum of \(r\) rank-one tensors.
- A rank-\(r\) tensor only contains essentially \(r \cdot \sum_{j=1}^{d} n_j\) pieces of information, which is potentially much smaller than the full dimension \(\prod_{j=1}^{d} n_j\).
- So a low-rank representation of \(\mathcal{T}\) is a sparse presentation.
Some Applications of Secant Varieties

- **Classical Algebraic Geometry:** When can a given projective variety $X \subset \mathbb{P}^n$ be isomorphically projected into $\mathbb{P}^{n-1}$?
  
  Determined by the dimension of the secant variety $\sigma_2(X)$.

- **Algebraic Complexity Theory:** Bound the border rank of algorithms via equations of secant varieties. Berkeley-Simons program Fall’14

- **Algebraic Statistics and Phylogenetics:**
  
  Given contingency tables for DNA of several species, determine the correct statistical model for their evolution.
  
  Find invariants (equations) of mixture models (secant varieties).
  
  For star trees / bifurcating trees this is the salmon conjecture.

- **Signal Processing:** Blind identification of under-determined mixtures, analogous to CDMA technology for cell phones.
  
  A given signal is the sum of many signals, one for each user.
  
  Decompose the signal uniquely to recover each user’s signal.

- **Computer Vision, Neuroscience, Quantum Information Theory, Chemistry...**
First algebraic / geometric questions for tensors

Let $X \subset \mathbb{P} \mathbb{C}^N$, with $N = n_1 \times \cdots \times n_d$, denote the set of rank-one tensors, and let $\sigma_r(X)$ denote the Zariski closure of the set of rank-$r$ tensors.

1. **[Dimensions]** What is the dimension of $\sigma_r(X)$?
   
   - When does $\sigma_r(X)$ fill the ambient $\mathbb{P} \mathbb{C}^N$? (defectivity)

2. **[Equations]** What are the polynomial defining equations of $\sigma_r(X)$?

3. **[Decomposition]** For my favorite $\mathcal{T} \in \mathbb{C}^N$, can you find an expression of $\mathcal{T}$ as a sum of points from $X$?

4. **[Specific Identifiability]** For a given $\mathcal{T} \in \mathbb{C}^N$, does $\mathcal{T}$ have a unique decomposition (ignoring trivialities)?

5. **[Generic Identifiability]** For *generic* $\mathcal{T} \in \mathbb{C}^N$, does $\mathcal{T}$ have a unique decomposition (ignoring trivialities)?

Today: Focus on Generic Identifiability
Geometric version of identifiability

Let $X \subset \mathbb{C}^N$, with $N = n_1 \times \cdots \times n_d$, denote the set of rank-one tensors, and let $\sigma_r(X)$ denote the Zariski closure of the set of rank-$r$ tensors. Construct the incidence variety

$$\mathcal{I} := \{([\mathcal{T}], [\mathcal{T}^1], \ldots, [\mathcal{T}^r]) \mid [\mathcal{T}^i] \in X, \mathcal{T} \in \langle \mathcal{T}^1, \ldots, \mathcal{T}^r \rangle \}$$

$$\mathcal{I} \subset \mathbb{P}^N \times X \times \ldots \times X$$

- Projection onto the first factor: $\pi(\mathcal{I}) = \sigma_r(X)$.
- Note $\dim(\mathcal{I}) = r \cdot \dim(\hat{X}) - 1$. If the fiber $\pi^{-1}([\mathcal{T}])$ over a generic $[\mathcal{T}] \in \sigma_r(X)$ is finite, then $\dim(\sigma_r(X)) = \dim(\mathcal{I})$.
- If $r$ is the smallest such that $\sigma_r(X) = \mathbb{P}^N$, say that $r$ is the generic rank.
- Moreover, $\#\pi^{-1}([\mathcal{T}])$ is the number of decompositions of $\mathcal{T}$.

**Definition**

*If $[\mathcal{T}] \in \sigma_r(X)$ is such that $\#\pi^{-1}([\mathcal{T}]) = r!$, then we say that $\mathcal{T}$ is identifiable and that the decomposition of $\mathcal{T}$ is essentially unique.*
Perfect identifiability for tensors

For $T \in \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$, based on dimension count, the generic rank is at least

$$R(n_1, \ldots, n_d) := \frac{\prod_{i=1}^{d} n_i}{\sum_{i=1}^{d} (n_i - 1) + 1} = \frac{\prod_{i=1}^{d} n_i}{\left(\sum_{i=1}^{d} n_i\right) + 1 - d}$$

- The value $\lceil R(n_1, \ldots, n_d) \rceil$ is called the expected generic rank.
- A necessary condition for generically finitely many decompositions is for $R(n_1, \ldots, n_d)$ to be an integer, a.k.a. perfect format.
- When the generic tensor of perfect format has an essentially unique decomposition, we say that perfect identifiability holds.
Known Results for “unbalanced formats”

Assume \( d \geq 3 \) and \( 2 \leq n_1 \leq n_2 \leq \ldots \leq n_d \). If \( n_d \geq \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1) \), we say that the format \((n_1, \ldots, n_d)\) is unbalanced.

Theorem (Catalisano-Geramita-Gimigliano’02, Abo-Ottaviani-Peterson’09, Bocci-Chiantini-Ottaviani’13)

For formats \((n_1, \ldots, n_d)\), suppose that \( n_d \geq \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1) \).

1. The generic rank is \( \min \left( n_d, \prod_{i=1}^{d-1} n_i \right) \).

2. A general tensor of rank \( r \) has a unique decomposition if \( r < \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1) \).

3. A general tensor of rank \( r = \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1) \) has exactly \( \binom{D}{r} \) different decompositions where \( D = \frac{\left( \sum_{i=1}^{d-1} (n_i-1) \right)!}{(n_1-1)! \cdots (n_{d-1}-1)!} \).

This value of \( r \) coincides with the generic rank in the perfect case: \( r = n_d \).

4. If \( n_d > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1) \), a general tensor of rank \( r > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1) \) has infinitely many decompositions.
Monodromy for Tensor Identifiability (using **Bertini**)

The problem:

Given $\mathcal{T}$ find rank-one tensors $\mathcal{T}^i$ so that $\mathcal{T} = \sum_{i=1}^{r} \mathcal{T}^i$.

Asks to solve a straightforward system of polynomial equations. In general, this can be a very difficult problem.

- One method to numerically solve large systems of polynomials is to use **homotopy continuation**, in a software package like **Bertini**.
- The idea is to start with a similar system $G$ whose solutions you know (like roots of unity). Then perform a homotopy to your system $F$:

  $$ t \cdot G + (1 - t) \cdot F \quad t \in [0, 1] $$

  and numerically track the paths traced out by the solutions of $G$. The paths should end in solutions of the $F$.

- *Generically* can construct paths that avoid singularities and end points are non-singular (real 1-dimensional path, singular locus has complex codim 1, so real codim 2.)
input: An affine variety $\mathcal{H}$.
Output: $\deg \mathcal{H}$

1. Choose a random linear space $\mathcal{L}$ with $\dim \mathcal{L} = \operatorname{codim} \mathcal{H}$.
2. Generate a point $x \in \mathcal{H} \cap \mathcal{L}$. Initialize $\mathcal{W} := \{x\}$.
3. Perform a random monodromy loop starting at the points in $\mathcal{W}$:
   (a) Pick a random loop $M(t)$ in the grassmannian of linear spaces so that $M(0) = M(1) = \mathcal{L}$.
   (b) Trace the curves $\mathcal{H} \cap M(t)$ starting at the points in $\mathcal{W}$ at $t = 0$ to compute the endpoints $\mathcal{E}$ at $t = 1$. (Hence, $\mathcal{E} \subset \mathcal{H} \cap \mathcal{L}$).
   (c) Update $\mathcal{W} := \mathcal{W} \cup \mathcal{E}$, sort $\mathcal{W}$, remove repeats and symmetric copies.
4. Repeat (2) until $\# \mathcal{W}$ stabilizes.
5. Use the trace test to verify that $\mathcal{W} = \mathcal{H} \cap \mathcal{L}$.
6. Return $\deg \mathcal{H} = \# \mathcal{H}(\cap \mathcal{L})$.

A triangular monodromy loop for random points $P_1$ and $P_2$ in $\mathbb{C}^N$:

$$
(\mathcal{L} \cap \mathcal{H}) \quad \longrightarrow \quad (\mathcal{L} + P_2) \cap \mathcal{H} \quad \leftarrow \quad (\mathcal{L} + P_1) \cap \mathcal{H}
$$
Monodromy for Tensor Decomposition (using Bertini)

Start: A general tensor $\mathcal{T}$ of format $(n_1, \ldots, n_d)$ with known minimal decomposition, $\mathcal{T} = \sum_{i=1}^{r}(v_1^i \otimes \ldots \otimes v_d^i)$.

(dehomogenize): Set $(v_j^i)_1 = 1$ for $i = 1, \ldots, r$ and $j = 1, \ldots, d - 1$.

- **Input system:**
  \[
  F_{\mathcal{T}}(v_1^1, \ldots, v_r^d) = \begin{bmatrix}
  \mathcal{T} - \sum_{i=1}^{r}(v_1^i \otimes \ldots \otimes v_d^i) \\
  (v_j^i)_1 - 1 \text{ for } i = 1, \ldots, r \text{ and } j = 1, \ldots, d - 1
  \end{bmatrix} = 0
  \]

- The system $F_{\mathcal{T}}$ consists of $\prod_{j=1}^{d} n_j + r(d - 1)$ polynomials in $r \cdot \sum_{j=1}^{d} n_j$ variables. Balanced format $\Rightarrow$ square system.

- Let $\mathcal{W} \subset (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})^r$ be the known decompositions of $\mathcal{T}$.

- Homotopy: For a loop $\tau : [0, 1] \rightarrow \mathbb{C}^{n_1 \cdots n_d}$ with $\tau(0) = \tau(1) = \mathcal{T}$, consider the homotopy
  \[
  H(v_1^1, \ldots, v_d^r, s) = F_{\tau(s)}(v_1^1, \ldots, v_d^r) = 0.
  \]

- Endpoints are decompositions of $\mathcal{T}$. If new, add results to $\mathcal{W}$.

- Repeat until $|\mathcal{W}|$ stabilizes (at least 20 additional randomly selected loops failed to yield any new decompositions), and possibly use AlphaCertify.
Theorem (CGG’02, AOP’09, BCC’13)

For formats \((n_1, \ldots, n_d)\), suppose that \(n_d \geq \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1)\).

1. The generic rank is \(\min\left(n_d, \prod_{i=1}^{d-1} n_i\right)\).

2. A general tensor of rank \(r\) has a unique decomposition if \(r < \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1)\).

3. A general tensor of rank \(r = \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1)\) has exactly \(\binom{D}{r}\) different decompositions where

\[
D = \frac{\left(\sum_{i=1}^{d-1} (n_i - 1)\right)!}{(n_1 - 1)! \cdots (n_{d-1} - 1)!}.
\]

This value of \(r\) coincides with the generic rank in the perfect case: when \(r = n_d\).

4. If \(n_d > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1)\), a general tensor of rank \(r > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1} (n_i - 1)\), e.g., a general tensor of format \((n_1, \ldots, n_d)\), has infinitely many decompositions.
## Computational results: Unbalanced cases

Some known perfect cases and the number of decompositions.

<table>
<thead>
<tr>
<th>$(n_1, \ldots, n_d)$</th>
<th>gen. rank</th>
<th># of decomp. of general tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, n, n)$</td>
<td>$n$</td>
<td>(Weierstrass-Kronecker) 1</td>
</tr>
<tr>
<td>$(3, 3, 5)$</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$(3, 4, 7)$</td>
<td>7</td>
<td>120</td>
</tr>
<tr>
<td>$(3, 5, 9)$</td>
<td>9</td>
<td>5005</td>
</tr>
<tr>
<td>$(3, 6, 11)$</td>
<td>11</td>
<td>352716</td>
</tr>
<tr>
<td>$(4, 4, 10)$</td>
<td>10</td>
<td>184756</td>
</tr>
<tr>
<td>$(2, 2, 2, 5)$</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$(2, 2, 3, 8)$</td>
<td>8</td>
<td>495</td>
</tr>
</tbody>
</table>
Computational results: Perfect cases, 3 factors

All perfect, balanced tensor formats of 3-tensors with $\prod_{i=1}^{3} n_i \leq 150$.

<table>
<thead>
<tr>
<th>$(n_1, n_2, n_3)$</th>
<th>gen. rank</th>
<th># of decomp. of general tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3, 4, 5)$</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$(3, 6, 7)$</td>
<td>9</td>
<td>38</td>
</tr>
<tr>
<td>$(4, 4, 6)$</td>
<td>8</td>
<td>62</td>
</tr>
<tr>
<td>$(4, 5, 7)$</td>
<td>10</td>
<td>$\geq 222,556$</td>
</tr>
</tbody>
</table>

After the numerical results, we were motivated to prove the following:

**Theorem (HOOS 2015)**

The general tensor of format $(3, 4, 5)$ has a unique decomposition as a sum of 6 decomposable summands.

Our proof relies on algebraic geometry, vector bundles and intersection theory, and relies on a notion of *non-abelian polarity*.
Computational results: Perfect cases, 4 factors

All perfect, balanced tensor formats with \( d \geq 4 \) and \( \prod_{i=1}^{d} n_i \leq 100 \).

<table>
<thead>
<tr>
<th>( (n_1, \ldots, n_d) )</th>
<th>gen. rank</th>
<th># of decomp. of general tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (2, 2, 2, 3) )</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( (2, 2, 3, 4) )</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>( (2, 2, 4, 5) )</td>
<td>8</td>
<td>68</td>
</tr>
<tr>
<td>( (2, 3, 3, 4) )</td>
<td>8</td>
<td>471</td>
</tr>
<tr>
<td>( (2, 3, 3, 5) )</td>
<td>9</td>
<td>7225</td>
</tr>
<tr>
<td>( (3, 3, 3, 3) )</td>
<td>9</td>
<td>20,596</td>
</tr>
<tr>
<td>( (2, 2, 2, 2, 4) )</td>
<td>8</td>
<td>447</td>
</tr>
<tr>
<td>( (2, 2, 2, 3, 3) )</td>
<td>9</td>
<td>18,854</td>
</tr>
<tr>
<td>( (2, 2, 2, 2, 2, 3) )</td>
<td>12</td>
<td>( \geq 238,879 )</td>
</tr>
</tbody>
</table>

Again, motivated by the numerical evidence we were able to prove:

**Theorem (HOOS 2015)**

*The general tensor of format \( (2, 2, 2, 3) \) has a unique decomposition as a sum of 4 decomposable summands.*

A similar proof to the \( (3, 4, 5) \)-case also works here.
A conjecture

Conjecture (HOOS 2015)

The only perfect formats \((n_1, \ldots, n_d)\) where a general tensor has a unique decomposition are

1. \((2, k, k)\) for some \(k\) – matrix pencils, known classically by Kronecker normal form,
2. \((3, 4, 5)\), and
3. \((2, 2, 2, 3)\).

The generic rank is known to be equal to the expected one for the cubic format \((n, n, n)\) [Lickteig’85], which is not perfect for \(n \geq 3\), and in the binary case \((2, \ldots, 2)\) for at least \(k \geq 5\) factors [CGG’11], which is perfect if \(k + 1\) is a power of 2. A numerical check for \(k = 7\) shows it is not identifiable.
Methods: Koszul Flattenings

The Koszul complex: linear maps $K_p: \Lambda^p V \to \Lambda^{p+1} V$ depending linearly on $V$.

$$K_p(v)(\varphi) = \varphi \wedge v \text{ for } p \geq 0, \quad K_p(v)(\varphi) = \varphi(v) \text{ for } p < 0.$$ 

Set $V_I = \Lambda^{i_1} V_1 \otimes \Lambda^{i_2} V_2 \otimes \cdots \otimes \Lambda^{i_d} V_d$, and form a tensor product of Koszul maps:

$$K_I: V_I \to V_{I+1}^d,$$

that depend linearly on $V_{(1,\ldots,1)} = V_1 \otimes \cdots \otimes V_d$.

Lemma (Koszul Flattening)

Suppose $T \in V_{1,\ldots,1}$ has tensor rank $r$. Let $i_j \geq 0$ for $j = 1, \ldots, h$, $i_j < 0$ for \( j = h + 1, \ldots, d \). The Koszul flattening $K_I(T): V_I \to V_{I+1}^d$ has rank at most

$$r_I := r \cdot \prod_{j=1}^{h} \binom{n_j - 1}{i_j} \cdot \prod_{j=h+1}^{d} \binom{n_j - 1}{-i_j - 1}.$$ 

In particular, the $(r_I + 1) \times (r_I + 1)$ minors of $K_I(T)$ vanish. Meaningful if $r_I < \min\{\dim V_I, \dim V_{I+1}^d\}$.

Basic idea: A Koszul flattening of $\mathcal{T}$ is a matrix constructed from the entries of $\mathcal{T}$ that has rank at most a multiple of the rank of $\mathcal{T}$: detect $\text{Rank}(\mathcal{T})$. 

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The $3 \times 4 \times 5$ case

Let us denote the three factors as $A = \mathbb{C}^3$, $B = \mathbb{C}^4$, $C = \mathbb{C}^5$. The following are all possible non-trivial, non-redundant Koszul flattenings (up to transpose).

- **usual flattenings:**
  \[ K_{(0,-1,-1)}: (B \otimes C)^* \rightarrow A, \]
  \[ K_{(-1,0,-1)}: (A \otimes C)^* \rightarrow B, \]
  \[ K_{(-1,-1,0)}: (A \otimes B)^* \rightarrow C, \]

- **Koszul flattenings:**
  \[ K_{(1,-1,0)}: B^* \otimes A \rightarrow C \otimes \bigwedge^2 A, \]
  \[ K_{(1,0,-1)}: C^* \otimes A \rightarrow B \otimes \bigwedge^2 A, \]
  \[ K_{(0,1,-1)}: C^* \otimes B \rightarrow A \otimes \bigwedge^2 B, \]
  \[ K_{(-1,1,0)}: A^* \otimes B \rightarrow C \otimes \bigwedge^2 B, \]
  \[ K_{(-1,0,1)}: A^* \otimes C \rightarrow B \otimes \bigwedge^2 C, \]
  \[ K_{(0,-1,1)}: B^* \otimes C \rightarrow A \otimes \bigwedge^2 C, \]
  \[ K_{(-1,0,2)}: A^* \otimes \bigwedge^2 C \rightarrow B \otimes \bigwedge^3 C, \]
  \[ K_{(0,-1,2)}: B^* \otimes \bigwedge^2 C \rightarrow A \otimes \bigwedge^3 C. \]
An example Koszul flattening

\[ K_{(0,1,-1)} : C^* \otimes B \to A \otimes \bigwedge^2 B \]

\( K_{0,1,-1}(a \otimes b \otimes c) \) has image

\[ (\bigwedge^0 A \wedge a) \otimes (\bigwedge^1 B \wedge b) \otimes (C^*(c)) \subset \bigwedge^1 A \otimes \bigwedge^2 B \otimes \bigwedge^0 C. \]

The factor \( C^*(c) \) is just a scalar that is obtained by contracting \( c \) with \( C^* \).

We are left with \( (\bigwedge^0 A \wedge a) = \langle a \rangle \) tensored with \( (\bigwedge^1 B \wedge b) \subset \bigwedge^2 B \),

but \( (\bigwedge^1 B \wedge b) \cong (B/b) \otimes \langle b \rangle \), which is 3 dimensional.

So \( K_{0,1,-1}(\mathcal{T}) \) has rank that is at most 3 times the rank of \( \mathcal{T} \). And since it is \( 18 \times 20 \), it has a chance to detect up to rank 6 tensors.
<table>
<thead>
<tr>
<th>map</th>
<th>size</th>
<th>mult-factor</th>
<th>max tensor rank detected</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{(0,-1,-1)}$</td>
<td>$3 \times 20$</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$K_{(-1,0,-1)}$</td>
<td>$4 \times 15$</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$K_{(-1,-1,0)}$</td>
<td>$5 \times 12$</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>$K_{(1,-1,0)}$</td>
<td>$15 \times 12$</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>$K_{(1,0,-1)}$</td>
<td>$12 \times 15$</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>$K_{(0,1,-1)}$</td>
<td>$18 \times 20$</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>$K_{(-1,1,0)}$</td>
<td>$12 \times 30$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$K_{(-1,0,1)}$</td>
<td>$40 \times 15$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$K_{(0,-1,1)}$</td>
<td>$30 \times 20$</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$K_{(-1,0,2)}$</td>
<td>$40 \times 30$</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>$K_{(0,-1,2)}$</td>
<td>$30 \times 40$</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>$K_{(0,-1,2)}$</td>
<td>$30 \times 40$</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

We see that the only maps that distinguish between tensor rank 5 and 6 are $K_{(1,-1,0)}$, $K_{(1,0,-1)}$, and $K_{(0,1,-1)}$. Since $\wedge^2 A \cong A^*$, the first two maps are transposes of each other:

$$K_{(1,-1,0)} = (K_{(1,0,-1)})^t.$$  

Thus, we proceed by considering $K_{(1,0,-1)}$ and $K_{(0,1,-1)}$. 

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Methods: Apolarity

The definition of the Koszul Flattening implies

\[ T = v_1 \otimes \ldots \otimes v_d \in \ker K_I(T) \iff \bigotimes_{j=1}^{h} (\varphi_j \wedge v_j) \otimes \bigotimes_{j=h+1}^{d} (\varphi_j(v_j)) = 0 \]

for all basis elements \( \varphi \in V_I \).

Think of elements of the kernel of \( K_I(T) \) as linear mappings.

Let \( N \sqcup P = \{ 1, \ldots, d \} \) be the set partition such that \(-I_N \in \mathbb{Z}^d_{>0}, I_P \in \mathbb{Z}^d_{\geq 0}\).

**Lemma (Non-abelian Apolarity Lemma [Landsberg-Ottaviani’13:])**

Suppose \( T = \sum_{s=1}^{r} v_1^s \otimes \ldots \otimes v_d^s \). The kernel \( \ker K_I(T) \) contains all maps \( \psi \in \text{Hom}(V_{-I_N}, V_{I_P}) \) such that

\[ \psi \left( V_{-I_N+1_N} \wedge \bigotimes_{j \in N} v_j^s \right) \wedge \left( \bigotimes_{j \in P} v_j^s \right) = 0 \]

for \( s = 1, \ldots, r \).

**Basic idea:** the kernel of a flattening of \( \mathcal{T} \) can be used to gain information about the decomposition of \( \mathcal{T} \).
In our case the Apolarity Lemma says that

$$\ker K_{1,0,-1}(\sum_{i=1}^{s} a_i b_i c_i) \supset \{ \varphi \in Hom(C, A) | \varphi(c_i) \wedge a_i = 0 \text{ for } i = 1, \ldots, s \}.$$  \hspace{1cm} (1)

and

$$\ker K_{0,1,-1}(\sum_{i=1}^{s} a_i b_i c_i) \supset \{ \varphi \in Hom(C, B) | \varphi(c_i) \wedge b_i = 0 \text{ for } i = 1, \ldots, s \}.$$  

Equality should hold for honest decompositions.

Basic result from Oeding-Ottaviani [OO’13] and Landsberg-Ottaviani [LO’11]:

The set of eigenvectors of a general element in $\ker(K_I(\mathcal{T}))$ (interpreted as the common base locus of general sections of a certain vector bundle) contains the set of (pieces of) rank-one summands in a decomposition of $\mathcal{T}$. 
Proof of Theorem 3-4-5: Vector Bundles

For general $\mathcal{T} \in A \otimes B \otimes C$, $K_{1,0,-1}(f)$ is surjective and $\ker K_{1,0,-1}(\mathcal{T})$ has dimension $\dim \text{Hom}(C, A) - \dim \wedge^2 A \otimes B = 15 - 12 = 3$.

Interpret $K_{1,0,-1}(\mathcal{T})$ as a map between sections of vector bundles.

Let $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ with (pull-back) line bundle. $\mathcal{O}(\alpha, \beta, \gamma)$

Let $Q_A$ be the pullback of the quotient bundle on $\mathbb{P}(A)$.

Let $E = Q_A \otimes \mathcal{O}(0, 0, 1)$ (a rank 2 bundle on $X$) and $L = \mathcal{O}(1, 1, 1)$.

As in [OO’13], [LO’13], the map $K_{1,0,-1}(\mathcal{T})$ can be identified with contraction $K_{1,0,-1}(\mathcal{T}): H^0(E) \rightarrow H^0(E^* \otimes L)^*$ depending linearly on $\mathcal{T} \in H^0(L)^*$.

Apolarity: compute the common base locus of the sections of the vector bundle $\ker(K_{0,1,-2}(\mathcal{T}))$ to find the decomposition of $\mathcal{T}$. 
Proof of Theorem 3-4-5: Intersection Theory I

- Have $K_{1,0,-1}(\mathcal{T}): H^0(E) \longrightarrow H^0(E^* \otimes L)^*$ depending linearly on $\mathcal{T} \in H^0(L)^*$.

- The general element in $H^0(E)$ vanishes on a codimension two subvariety of $X$ which has homology class $c_2(E) \in H^*(X, \mathbb{Z})$.

- The ring $H^*(X, \mathbb{Z})$ can be identified with $\mathbb{Z}[t_A, t_B, t_C]/(t_A^3, t_B^4, t_C^5)$.

- The Chern polynomial of $Q_A$ is $\frac{1}{1+t_A}$, so $c_2(E) = t_A^2 + t_A t_C + t_C^2$.

- Three general sections of $H^0(E)$ have common base locus given by $c_2(E)^3 = (t_A^2 + t_A t_C + t_C^2)^3 = 6t_A^2 t_C^4$.

- This coefficient 6 coincides with the generic rank and it is the key to the computation.
Proof of Theorem 3-4-5: Intersection Theory II

- A Macaulay2 test (M2 file on arXiv) performed on a random tensor $\mathcal{T}$ gives that the common base locus of $\ker K_{1,0,-1}(\mathcal{T})$ is given by 6 points $(a_i, c_i)$ for $i = 1, \ldots, 6$ on the 2-factor Segre variety $\mathbb{P}(A) \times \mathbb{P}(C)$.

- By semicontinuity, the common base locus of $\ker K_{1,0,-1}(\mathcal{T})$ is given by 6 points for general tensor $\mathcal{T}$. Hence, for the general tensor $\mathcal{T}$, equality holds in the Apolarity Lemma.

- In particular, the decomposition $\mathcal{T} = \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i$ has a unique solution (up to scalar) for $a_i, c_i$. It follows that also the remaining vectors $b_i$ can be recovered uniquely, by solving a linear system.
Thanks!
The $2 \times 2 \times 2 \times 3$ case

For this part, let $A \cong B \cong C \cong \mathbb{C}^2$ and $D \cong \mathbb{C}^3$. The only interesting Koszul flattenings for tensors in $A \otimes B \otimes C \otimes D$ are the following maps, which depend linearly on $A \otimes B \otimes C \otimes D$.

The 1-flattenings (and their transposes):

- $K_{-1,0,0,0} : A^* \rightarrow B \otimes C \otimes D$, $K_{0,-1,0,0} : B^* \rightarrow A \otimes C \otimes D$,
- $K_{0,0,-1,0} : C^* \rightarrow A \otimes B \otimes D$, $K_{0,0,0,-1} : D^* \rightarrow A \otimes B \otimes C$,

which detect a maximum of rank 2 in the first 3 cases and a maximum of rank 3 in the last.

The 2-flattenings (and their transposes):

- $K_{0,0,-1,-1} : C^* \otimes D^* \rightarrow A \otimes B$, $K_{0,-1,0,-1} : B^* \otimes D^* \rightarrow A \otimes C$,
- $K_{-1,0,0,-1} : A^* \otimes D^* \rightarrow B \otimes C$.

The maps are all $4 \times 6$ and detect a maximum of tensor rank 4.

The higher Koszul flattenings:

- $K_{-1,0,0,1} : A^* \otimes D \rightarrow B \otimes C \otimes \Lambda^2 D$, $K_{0,-1,0,1} : B^* \otimes C \rightarrow A \otimes C \otimes \Lambda^2 D$,
- $K_{0,0,-1,1} : C^* \otimes D \rightarrow A \otimes B \otimes \Lambda^2 D$.

These maps are all $12 \times 6$, and detect a maximum of rank 3.
Proof of Theorem 2-2-2-3

Suppose $T \in A \otimes B \otimes C \otimes D$. Consider $K_{0,0,-1,-1}: C^* \otimes D^* \to A \otimes B$. If $T$ is general of rank 4, then Rank $K_{0,0,-1,-1}(T) = 4$ and dim ker $K_{0,0,-1,-1}(T) = 2$. Apolarity says that the points $\{c^s \otimes d^s\}$ are in the common base locus of the elements in the kernel of $K_{0,0,-1,-1}(T)$.

Consider line bundles $E = \mathcal{O}(0, 0, 1, 1)$, $L = \mathcal{O}(1, 1, 1, 1)$ over Seg($\mathbb{P}C^* \times \mathbb{P}D^*$). Two general sections of $E$ have common base locus given by a cubic curve, denoted $C_{C,D}$ of bi-degree (1,2) on Seg($\mathbb{P}C \times \mathbb{P}D$). The projection to $\mathbb{P}D$ is a conic, which we denote $Q_C$.

Repeat the process for the next 2-flattening, $K_{0,-1,0,-1}: B^* \otimes D^* \to A \otimes C$,

changing the roles of $C$ and $B$, we obtain another conic $Q_B$ in $\mathbb{P}D^*$.

Finally, if $Q_C$ and $Q_B$ are general, Bézout’s theorem implies that they intersect in 4 points in $\mathbb{P}D$, $\{[d^1], [d^2], [d^3], [d^4]\}$.

Pull back the $\{d_i\}$ to the curve $C_{C,D}$ in Seg($\mathbb{P}C^* \times \mathbb{P}D^*$) and project to $\mathbb{P}C$ to obtain 4 points $\{c_i\}$ on $\mathbb{P}C$.

Reverse the roles of $B$ and $C$ and repeat to find 4 points $\{b_i\}$ on $\mathbb{P}B$.

Reverse the roles of $A$ and $B$ and repeat to find 4 points $\{a_i\}$ on $\mathbb{P}A$.

The tensor products $a^i \otimes b^i \otimes c^i \otimes d^i$ obtained in this way are, up to scale, the indecomposable tensors in the decomposition of the original tensor $T$.

Finally we solve an easy linear system to determine the coefficients $\lambda_i$ in the expression $T = \sum_{i=1}^{4} \lambda_i a^i \otimes b^i \otimes c^i \otimes d^i$. 