Annotated Examples 6

Greuel, Laplagne, Seelich

$$y^9 + y^8x + yx^8 + x^9 + y^8 + y^5 + y^4x + y^3x^2 + y^2x^3 = 0$$

I don't know the origin or reason to consider this either, but it is example I_2 in Greuel, Laplagne, Seelisch. It does reduce to a disguised Hermitian curve in characteristic 2 however if one uses a new variable $z := x + y$; but, in general, it has nine points at infinity, that is, nine points at which both rational functions y and x have (simple) poles.

Running this in SINGULAR 3-1-0 in characteristic 0 gives the obvious results:

```
> ring r=0,(y,x),dp;
> ideal i=y9+y8x+y8+y5+y4x+y3x2+y2x3+yx8+x9;
> list nor=normal(i);
> nor;
[1]:
   [1]:
//   characteristic : 0
//   number of vars : 6
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3) T(4)
//        block   2 : ordering dp
//                  : names    y x
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=22y3x3+17y2x4+12yx5+9x6
      _[2]=286661055248x8+87898245974y2x5-12353330949yx6+62048493297x7-197090663616y3x+12038
      _[3]=286661055248yx7-80274702838y2x5+14103995493yx6-35958373353x7+272680125344y3x+3633
      _[4]=286661055248y2x6+76134447686y2x5+7827828195yx6+32839651161x7-207074171392y3x-6810
      _[5]=22y4+17y3x+12y2x2+9yx3
> def R=nor[1][1];
// ** redefining R **
> setring R;
> normap;
normap[1]=y
normap[2]=x
> norid;
norid[1]=3648850129*T(2)*y+3995374817*T(3)*y-3648850129*T(3)*x-3995374817*T(4)*x-16832046024
norid[2]=3995374817*T(1)*x^2-T(2)*y+T(3)*x-8958666528*x
norid[3]=286661055248*T(2)*y*x+76134447686*T(2)*y-35958373353*T(2)*x-286661055248*T(3)*x^2-
norid[4]=-T(1)*y+x^3
norid[5]=8131378854333494670372677539134764 8*T(1)*y^3+9490012318020946459389491913245237 44*T
```

```
norid[6]=3327421222123933065968198494598463846999202689066245698155526600151250608619056675
norid[7]=3218599534830757703361149501391524486417840698000236084799778317468887763571921098
norid[8]=3808255130662774689132902585764942809156916763637801197716567211756208206693366034
norid[9]=T(1)^2-68200714730263703/221690074730412816*T(1)*y^2+71391067772774521948965332 3/3
norid[10]=T(1)*T(2)+2795072098940431/234256*T(1)*y^4-2896734265060801 1/5153632*T(1)*y^3*x+5
03943435058803916962774586[illegible]
norid[12]=T(1)*T(3)-30449487904648867868795508943/665070224191238448*T(1)*y^4+3219867321906
627566741427979648407019360[illegible]
740810933836876029281705434[illegible]
norid[15]=T(1)*T(4)+34795516414487602791286854793/665070224191238448*T(1)*y^4-3004062727886
97926526680350077808757690[illegible]
7491866976256*T(4)*y-27904[illegible]
933179491/1045831291807828[illegible]
norid[19]=y^9+y^8*x+y*x^8+x^9+y^8+y^5+y^4*x+y^3*x^2+y^2*x^3
```

or the Gröbner basis version

```
> option(redSB);
> ideal j=std(norid);j;
j[1]=y^9+y^8*x+y*x^8+x^9+y^8+y^5+y^4*x+y^3*x^2+y^2*x^3
j[2]=1692570593161670464785746419272*T(4)*y*x^2+1702467834074976605397238464648*T(4)*x^3+24
j[3]=1692570593161670464785746419272*T(4)*y^2*x-1077282969443214301544208589704*T(4)*x^3-15
j[4]=1692570593161670464785746419272*T(4)*y^3+5962423550040890020252910101 20*T(4)*x^3+84740
j[5]=1692570593161670464785746419272*T(4)*x^4+1918959204357138030096381727 44*T(4)*x^3+27272
j[6]=22478884260612007950397182 21*T(3)*y+4912449051422523011063670 09*T(3)*x-881379387578481
j[7]=1338834954540289881*T(3)*x^2+914093595084379303*T(4)*y^2+719156740803785063*T(4)*y*x+1
j[8]=745655270310092680261416217347227319*T(2)*x+616422587367327207956677997187951666*T(3)*
j[9]=820220797341101948287557839081950050 9*T(2)*y-101649154963959281293762583769377128 62*T(
j[10]=T(1)*y-x^3
j[11]=820220797341101948287557839081950050 9*T(1)*x^2-491244905142252301106367009*T(3)*x+881
j[12]=76604557686364050060372264754510095933462002317681121124467383488*T(4)^2+134575759571
j[13]=24711147640762596793668472501454869655955484618606813265957 22048*T(3)*T(4)-556182170
j[14]=53623190380454835042260585328157067153423401622376784787127 1684416*T(2)*T(4)+16151470
j[15]=11756267965107811854963002835003318857031065606 16*T(1)*T(4)+145713664650147725231130
j[16]=17874396793484945014086861776052355717807800540792261595709 0561472*T(3)^2+52708773454
j[17]=17874396793484945014086861776052355717807800540792261595709 0561472*T(2)*T(3)-64830979
j[18]=391875598836927061832100094500110628567702186872*T(1)*T(3)-35106828104403341992126214
j[19]=595813226449498167136228725868411857260260018026408719856968 53824*T(2)^2+263006658315
j[20]=11756267965107811854963002835003318857031065606 16*T(1)*T(2)+135681440027002565622223 9
j[21]=88861635552146603851051145665626942112771462896274840576*T(1)^2+11240351199219834596 3
```

Compare this to MAGMA's

```
Q:=Rationals();
FF<x>:=FunctionField(Q);
P<y>:=PolynomialRing(FF);
f:=y^9+y^8*x+y*x^8+x^9+y^8+y^5+y^4*x+y^3*x^2+y^2*x^3;
```

```
Ff<Y>:=RationalExtensionRepresentation(FunctionField(f));
C<X>:=CoefficientRing(Ff);
INT:=Integers(C);
IC:=IntegralClosure(INT,Ff);
B:=Basis(IC);
for i in [1..#B] do i,B[i]; end for;
Genus(Ff);

1 1
2 Y
3 Y^2
4 Y^3
5 Y^4
6 1/X*Y^5 + 1/X*Y^4 + 1/X*Y
7 1/X^2*Y^6 + 1/X^2*Y^5 - 1/X*Y^4 + 1/X^2*Y^2
8 1/X^3*Y^7 + 1/X^3*Y^6 - 1/X^2*Y^5 + 1/X*Y^4 + 1/X^3*Y^3 + 1/X*Y
9 1/X^6*Y^8 + (X + 1)/X^6*Y^7 + 1/X^6*Y^4 + 1/X^5*Y^3 + 1/X^4*Y^2 + 1/X^3*Y
16
```

which gives a predictable module basis with 9 elements, four being necessary to define the integral closure over the original ring, but all nine to define it as an *algebra* ver $\mathbf{Q}[x]$.

My qth-power algorithm will actually work on this, though it tries to label both y and x as f_1 in the output. But this can be salvaged when I rework the code.

```
Magma V2.14-15    Thu May 21 2009 10:52:31 on cactus   [Seed = 3265130547]
Type ? for help.  Type <Ctrl>-D to quit.
> load "char0_input.mag";
Loading "char0_input.mag"
Loading "closure_func051709.mag"
555555555555555555555555555555555555555555555555
666666666666666666666666666666666666666666666666
> load "char0_function.mag";
Loading "char0_function.mag"
11111111111111111111111111111111111111111111111111111111
q= 3
Delta= $.2^9
WT_MATRIX_T= [
    [ 4, 4, 4, 4, 3, 2, 2, 1, 1 ]
]
time for q= 3 is 0.380 seconds
modulus= 3
11111111111111111111111111111111111111111111111111111111
q= 5
Delta= $.2^9
WT_MATRIX_T= [
```

```
    [ 4, 4, 4, 4, 3, 2, 2, 1, 1 ]
]
time for q= 5 is 0.670 seconds
modulus= 15
[
    f_1^7*f_1^3 + f_1^6*f_1^3 - f_1^5*f_1^4 + f_1^4*f_1^5 + f_1^3*f_1^3 +
        f_1*f_1^5,
    f_1^6*f_1^4 + f_1^5*f_1^4 - f_1^4*f_1^5 + f_1^2*f_1^4,
    f_1^5*f_1^5 + f_1^4*f_1^5 + f_1*f_1^5,
    f_1^4*f_1^6,
    f_1^3*f_1^6,
    f_1^8 + f_1^7*f_1 + f_1^7 + f_1^4 + f_1^3*f_1 + f_1^2*f_1^2 + f_1*f_1^3,
    f_1^2*f_1^6,
    f_1*f_1^6,
    f_1^7,
    f_1^6
]
1 f_1^7*f_1^3 + f_1^6*f_1^3 - f_1^5*f_1^4 + f_1^4*f_1^5 + f_1^3*f_1^3 +
    f_1*f_1^5
2 f_1^6*f_1^4 + f_1^5*f_1^4 - f_1^4*f_1^5 + f_1^2*f_1^4
3 f_1^5*f_1^5 + f_1^4*f_1^5 + f_1*f_1^5
4 f_1^4*f_1^6
5 f_1^3*f_1^6
6 f_1^8 + f_1^7*f_1 + f_1^7 + f_1^4 + f_1^3*f_1 + f_1^2*f_1^2 + f_1*f_1^3
7 f_1^2*f_1^6
8 f_1*f_1^6
9 f_1^7
10 f_1^6
newrelations= [
    f_1^2 - f_2,
    f_2^2 - f_4,
    f_2*f_1 - f_3,
    f_2^2 + f_4 + f_4 + f_2*f_1^2 + 1,
    f_2*f_2 + f_2*f_1^2 + f_1*f_1^3,
    f_2*f_1 + f_1*f_1^2 + f_1^3,
    f_3^2 - f_4*f_1^2 + f_4*f_1 - f_4*f_1 - f_4 + f_2 - f_1,
    f_3*f_2 + f_3*f_1^2 + f_2*f_1^3,
    f_3*f_2 - f_4*f_1 + f_4 + f_1,
    f_3*f_1 - f_4,
    f_4^2 + f_4*f_1^4 - f_2*f_1^6 + f_4*f_1^3 - f_4*f_1^3 + f_4*f_1^3 -
        f_4*f_1^3 - f_4*f_1^2 + 2*f_4*f_1^2 - 3*f_4*f_1^2 + f_4*f_1 - 3*f_4*f_1
        + f_2*f_1^2 - f_3 + f_2*f_1 - 2*f_1*f_1^2 + f_2 - 2*f_1*f_1 - f_1,
    f_4*f_3 - f_4*f_1^3 + f_4*f_1^2 - f_4*f_1^2 + f_4*f_1^2 - f_4*f_1 +
        2*f_4*f_1 + f_4 + f_3 + f_1*f_1^2 - f_2 + f_1*f_1 + f_1,
    f_4*f_2 + f_4*f_1^2 + f_3*f_1^3,
    f_4*f_2 - f_4*f_1^2 + f_4*f_1 - f_4*f_1 - f_4 + f_2 - f_1,
```

```
f_4*f_1 - f_4*f_1 + f_4 + f_1,
f_4^2 + f_4*f_1^4 - f_2*f_1^6 + f_2*f_1^6 - f_1^8 + f_4*f_1^3 - f_4*f_1^3 +
    f_4*f_1^3 - f_4*f_1^3 + f_1*f_1^6 + f_1^7 - f_4*f_1^2 + 2*f_4*f_1^2 -
    3*f_4*f_1^2 + f_4*f_1 - 3*f_4*f_1 - f_4 + 2*f_2*f_1 - 2*f_1*f_1^2 + f_2
    - 2*f_1*f_1 - f_1,
f_4*f_4 - f_4*f_1^4 + f_2*f_1^6 + f_1*f_1^7 + f_1^8 - f_4*f_1^3 + f_4*f_1^3
    - f_4*f_1^3 + f_4*f_1^3 + f_4*f_1^2 - 2*f_4*f_1^2 + 3*f_4*f_1^2 -
    f_4*f_1 + 3*f_4*f_1 + f_4 + f_3*f_1 + f_3 - f_2*f_1 + 2*f_1*f_1^2 - f_2
    + 2*f_1*f_1 + f_1,
f_4*f_3 + f_4*f_1^3 - f_2*f_1^5 - f_4*f_1^2 + f_4*f_1^2 - f_4*f_1^2 +
    f_4*f_1 - 2*f_4*f_1 - f_4 + f_2*f_1 + f_2 - f_1*f_1 - f_1,
f_4*f_2 + f_4*f_1^2 + f_4*f_1^2 + f_3*f_1^2 + f_1^2,
f_4*f_2 - f_4*f_1^2 - f_4*f_1 + f_4*f_1 + f_4 + f_1*f_1 + f_1,
f_4*f_1 - f_4*f_1 - f_4,
f_4^2 + f_4*f_1^4 + f_4*f_1^4 - f_2*f_1^6 - f_1^8 + f_4*f_1^3 - f_4*f_1^3 +
    f_4*f_1^3 - f_4*f_1^3 + f_3*f_1^4 - f_2*f_1^5 - f_1*f_1^6 + f_1^7 -
    f_4*f_1^2 + 2*f_4*f_1^2 - 3*f_4*f_1^2 + f_4*f_1 - 3*f_4*f_1 + f_4 + f_4
    - f_4 - f_2*f_1^2 - 2*f_3 + 2*f_2*f_1 - 2*f_1*f_1^2 + f_2 - 2*f_1*f_1 -
    f_1,
f_4*f_4 - f_4*f_1^4 + f_3*f_1^5 + f_2*f_1^6 + f_1^8 - f_4*f_1^3 + f_4*f_1^3
    - f_4*f_1^3 + f_4*f_1^3 + f_2*f_1^5 - f_1^7 + f_4*f_1^2 - 2*f_4*f_1^2 +
    3*f_4*f_1^2 - f_4*f_1 + 3*f_4*f_1 - f_4 + f_4 + f_4 + f_3 - 2*f_2*f_1 +
    2*f_1*f_1^2 - f_2 + 2*f_1*f_1 + f_1,
f_4*f_4 + f_4*f_1^4 - f_2*f_1^6 + f_2*f_1^6 - f_1^8 + f_4*f_1^3 - f_4*f_1^3
    + f_4*f_1^3 - f_4*f_1^3 - f_4*f_1^2 + 2*f_4*f_1^2 - 3*f_4*f_1^2 +
    f_4*f_1 - 3*f_4*f_1 - f_3 + f_2*f_1 - 2*f_1*f_1^2 + f_2 - 2*f_1*f_1 -
    f_1,
f_4*f_3 - f_4*f_1^3 + f_2*f_1^5 + f_1*f_1^6 + f_1^7 + f_4*f_1^2 - f_4*f_1^2
    + f_4*f_1^2 - f_4*f_1 + 2*f_4*f_1 + f_4 + f_3 - f_2 + f_1*f_1 + f_1,
f_4*f_2 + f_4*f_1^2 + f_4*f_1^2 - f_3*f_1^2,
f_4*f_2 + f_4*f_1^2 - f_2*f_1^4 + f_4*f_1 - f_4*f_1 - f_4 + f_2 - f_1,
f_4*f_1 - f_4*f_1 + f_4 + f_1,
f_4^2 + f_4*f_1^4 + f_4*f_1^4 - f_2*f_1^6 - f_1^8 + f_4*f_1^3 - f_4*f_1^3 +
    f_4*f_1^3 - f_4*f_1^3 + f_3*f_1^4 + f_2*f_1^5 - f_1*f_1^6 + f_1^7 -
    f_4*f_1^2 + 2*f_4*f_1^2 - 3*f_4*f_1^2 + f_4*f_1 - 3*f_4*f_1 + 2*f_4 -
    f_4 - 2*f_2*f_1^2 - f_1^4 - 2*f_3 + 2*f_2*f_1 - 2*f_1*f_1^2 + f_2 -
    2*f_1*f_1 - f_1,
f_4*f_4 - f_4*f_1^4 + f_4*f_1^4 + f_2*f_1^6 + f_1^8 - f_4*f_1^3 + f_4*f_1^3
    - f_4*f_1^3 + f_4*f_1^3 - f_3*f_1^4 + f_1*f_1^6 - f_1^7 + f_4*f_1^2 -
    2*f_4*f_1^2 + 3*f_4*f_1^2 - f_4*f_1 + 3*f_4*f_1 - f_4 + f_4 + f_2*f_1^2
    + f_1^4 + 2*f_3 - 2*f_2*f_1 + 2*f_1*f_1^2 - f_2 + 2*f_1*f_1 + f_1,
f_4*f_4 + f_4*f_1^4 + f_4*f_1^4 - f_2*f_1^6 - f_1^8 + f_4*f_1^3 - f_4*f_1^3
    + f_4*f_1^3 - f_4*f_1^3 + f_3*f_1^4 + f_1^7 - f_4*f_1^2 + 2*f_4*f_1^2 -
    3*f_4*f_1^2 + f_4*f_1 - 3*f_4*f_1 + f_4 - f_4 - f_2*f_1^2 - f_3 +
    2*f_2*f_1 - 2*f_1*f_1^2 + f_2 - 2*f_1*f_1 - f_1,
f_4*f_4 - f_4*f_1^4 + f_3*f_1^5 + f_2*f_1^6 + f_1^8 - f_4*f_1^3 + f_4*f_1^3
```

```
        - f_4*f_1^3 + f_4*f_1^3 + f_4*f_1^2 - 2*f_4*f_1^2 + 3*f_4*f_1^2 -
        f_4*f_1 + 3*f_4*f_1 + f_4 + f_3 - f_2*f_1 + 2*f_1*f_1^2 - f_2 +
        2*f_1*f_1 + f_1,
    f_4*f_3 + f_4*f_1^3 - f_2*f_1^5 + f_2*f_1^5 - f_1^7 - f_4*f_1^2 + f_4*f_1^2
        - f_4*f_1^2 + f_4*f_1 - 2*f_4*f_1 - f_4 + f_2 - f_1*f_1 - f_1,
    f_4*f_2 + f_4*f_1^2 + f_4*f_1^2 + f_3*f_1^2 + f_1^2,
    f_4*f_2 - f_4*f_1^2 + f_2*f_1^4 + f_1*f_1^5 + f_1^6 - f_4*f_1 + f_4*f_1 +
        f_4 + f_1,
    f_4*f_1 + f_4*f_1 - f_2*f_1^3 - f_4
]
totaltime= 1.130 seconds
```

At least one can see how the fractions printed before the newrealtions correspond to what MAGMA's IntegralClosure above produced, by sorting out which f_1 is which.

Macaulay2 is way too slow on this. I will repost when it actually gives me results, even for icFractions.

```
Macaulay 2, version 1.2
with packages: Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, SchurRings,
               TangentCone

i1 : R=QQ[y,x]

o1 = R

o1 : PolynomialRing

i2 : I=ideal(y^9+y^8*x+y*x^8+x^9+y^8+y^5+y^4*x+y^3*x^2+y^2*x^3)

             9    8       8    9    8    5    4     3 2    2 3
o2 = ideal(y  + y x + y*x  + x  + y  + y  + y x + y x  + y x )

o2 : Ideal of R

i3 : S=R/I

o3 = S

o3 : QuotientRing

i4 : F=icFractions(S)
```

Again MAGMA's Normalisation,

```
t:=Cputime();
```

```
SetLogFile("may2809");
F:=Rationals();
P<y,x>:=PolynomialRing(F,2);

f:=y^9+y^8*x+y*x^8+x^9+y^8+y^5+y^4*x+y^3*x^2+y^2*x^3;
I:=ideal<P|f>;
N:=Normalisation(I);
J:=N[1][1];J;
"Normalisation time=",Cputime(t);
```

gives readable output

```
$.1^3 - $.2*$.3,
$.1^3*$.3 - $.2*$.3^2,
$.1*$.3^2 + $.1 - $.2*$.5,
$.1*$.5 - $.2*$.4,
$.1*$.2^4 + $.1*$.4 + $.1 + $.2^5 + $.2^4 + $.2*$.4 + $.2,
$.1^4 - $.1*$.2*$.3,
$.1^4*$.3 + $.1*$.2 - $.2^2*$.5,
$.1^2*$.3^2 + $.1^2 - $.2^2*$.4,
$.1^2*$.5 + $.1*$.2^5 + $.1*$.2 + $.2^6 + $.2^5 + $.2^2*$.4 + $.2^2,
$.1^2*$.2^4 + $.1^2*$.4 + $.1^2 + $.1*$.2^4 - $.2^6 - $.2^5 - $.2^2*$.4 -
    $.2^2,
$.1*$.2^5 + $.1*$.2 + $.2^6 + $.2^5 + $.2^2*$.4 + $.2^2 + $.3^3 + $.3,
$.1^2*$.2^4 + $.1^2 + $.1*$.2^4 - $.2^6 - $.2^5 - $.2^2*$.4 - $.2^2 +
    $.3*$.5,
$.1^2*$.2^3 - $.1*$.2^4 + $.2^6 + $.2^5 + $.2^4*$.3 + $.2^2*$.4 + $.2^2 +
    $.3*$.4 + $.3,
$.1^3*$.3^2 + $.1*$.2^6 + $.1*$.2^2 + $.2^7 + $.2^6 + $.2^3*$.4 + $.2^3 +
    $.2*$.3,
$.1^2*$.2^5 + $.1^2*$.2 + $.1*$.2^5 + $.1*$.3^3 + $.1*$.3 - $.2^7 - $.2^6 -
    $.2^3*$.4 - $.2^3,
$.1^2*$.2^4 - $.1*$.2^5 + $.1*$.3*$.5 + $.2^7 + $.2^6 + $.2^5*$.3 +
    $.2^3*$.4 + $.2^3 + $.2*$.3,
-$.1^2*$.2^4 + $.1*$.2^5 + $.1*$.2^4*$.3 + $.1*$.3*$.4 + $.1*$.3 - $.2^7 -
    $.2^6 + $.2^4*$.3 - $.2^3*$.4 - $.2^3,
$.1^4*$.3^2 + $.1^2*$.2^6 + $.1^2*$.2^2 + $.1*$.2^6 + $.1*$.2*$.3 - $.2^8 -
    $.2^7 - $.2^4*$.4 - $.2^4,
$.1^2*$.2^5 + $.1^2*$.3^3 + $.1^2*$.3 - $.1*$.2^6 + $.2^8 + $.2^7 +
    $.2^6*$.3 + $.2^4*$.4 + $.2^4 + $.2^2*$.3,
-$.1^2*$.2^5 + $.1^2*$.3*$.5 + $.1*$.2^6 + $.1*$.2^5*$.3 + $.1*$.2*$.3 -
    $.2^8 - $.2^7 + $.2^5*$.3 - $.2^4*$.4 - $.2^4,
$.1^2*$.2^5 + $.1^2*$.2^4*$.3 + $.1^2*$.3*$.4 + $.1^2*$.3 - $.1*$.2^6 +
    $.1*$.2^4*$.3 + $.2^8 + $.2^7 - $.2^5*$.3 + $.2^4*$.4 + $.2^4,
-$.1^2*$.2^5 + $.1*$.2^6 + $.1*$.2^5*$.3 + $.1*$.2*$.3 - $.2^8 - $.2^7 +
    $.2^5*$.3 - $.2^4*$.4 - $.2^4 + $.3^4 + $.3^2,
$.1^2*$.2^5 + $.1^2*$.2^4*$.3 + $.1^2*$.3 - $.1*$.2^6 + $.1*$.2^4*$.3 +
```

```
        $.2^8 + $.2^7 - $.2^5*$.3 + $.2^4*$.4 + $.2^4 + $.3^2*$.5,
    -$.1^2*$.2^5 + $.1^2*$.2^3*$.3 + $.1*$.2^6 - $.1*$.2^4*$.3 - $.2^8 - $.2^7 +        $.2
    -$.1^2*$.2^5 + $.1^2*$.2^3*$.3 + $.1*$.2^6 - $.1*$.2^4*$.3 - $.2^8 - $.2^7 +        $.2
    $.1^2*$.2^5 - $.1^2*$.2^3*$.3 - $.1*$.2^6 + $.1*$.2^4*$.3 + $.2^8 + $.2^7 -
        $.2^5*$.3 + $.2^4*$.4 + $.2^4*$.5 + 2*$.2^4 + $.2^3*$.3^2 + $.2^3 +
        $.4*$.5 + $.4 + $.5 + 1,
    -$.1^2*$.2^5 + $.1^2*$.2^3*$.3 + $.1*$.2^6 - $.1*$.2^4*$.3 - $.2^8 - $.2^7 +        $.2
```

its Groebner basis is a mess, but reversing the order of the variables

```
t:=Cputime();
F:=Rationals();
P<e,d,c,b,a>:=PolynomialRing(F,5,"lex");


f1:=a^3 - b*c;
f2:=a^3*c - b*c^2;
f3:=a*c^2 + a - b*e;
f4:=a*e - b*d;
f5:=a*b^4 + a*d + a + b^5 + b^4 + b*d + b;
f6:=a^4 - a*b*c;
f7:=a^4*c + a*b - b^2*e;
f8:=a^2*c^2 + a^2 - b^2*d;
f9:=a^2*e + a*b^5 + a*b + b^6 + b^5 + b^2*d + b^2;
f10:=a^2*b^4 + a^2*d + a^2 + a*b^4 - b^6 - b^5 - b^2*d - b^2;
f11:=a*b^5 + a*b + b^6 + b^5 + b^2*d + b^2 + c^3 + c;
f12:=a^2*b^4 + a^2 + a*b^4 - b^6 - b^5 - b^2*d - b^2 + c*e;
f13:=a^2*b^3 - a*b^4 + b^6 + b^5 + b^4*c + b^2*d + b^2 + c*d + c;
f14:=a^3*c^2 + a*b^6 + a*b^2 + b^7 + b^6 + b^3*d + b^3 + b*c;
f15:=a^2*b^5 + a^2*b + a*b^5 + a*c^3 + a*c - b^7 - b^6 - b^3*d - b^3;
f16:=a^2*b^4 - a*b^5 + a*c*e + b^7 + b^6 + b^5*c + b^3*d + b^3 + b*c;
f17:=-a^2*b^4 + a*b^5 + a*b^4*c + a*c*d + a*c - b^7 - b^6 + b^4*c - b^3*d - b^3;
f18:=a^4*c^2 + a^2*b^6 + a^2*b^2 + a*b^6 + a*b*c - b^8 - b^7 - b^4*d - b^4;
f19:=a^2*b^5 + a^2*c^3 + a^2*c - a*b^6 + b^8 + b^7 +  b^6*c + b^4*d + b^4 + b^2*c;
f20:=-a^2*b^5 + a^2*c*e + a*b^6 + a*b^5*c + a*b*c - b^8 - b^7 + b^5*c - b^4*d - b^4;
f21:=a^2*b^5 + a^2*b^4*c + a^2*c*d + a^2*c - a*b^6 + a*b^4*c + b^8 + b^7 - b^5*c + b^4*d + b
f22:=-a^2*b^5 + a*b^6 + a*b^5*c + a*b*c - b^8 - b^7 + b^5*c - b^4*d - b^4 + c^4 + c^2;
f23:=a^2*b^5 + a^2*b^4*c + a^2*c - a*b^6 + a*b^4*c + b^8 + b^7 - b^5*c + b^4*d + b^4 + c^2*e
f24:=-a^2*b^5 + a^2*b^3*c + a*b^6 - a*b^4*c - b^8 - b^7 + b^5*c + b^4*c^2 - b^4*d - b^4 + c
f25:=-a^2*b^5 + a^2*b^3*c + a*b^6 - a*b^4*c - b^8 - b^7 + b^5*c + b^4*c^2 - b^4*d - b^4 + c
f26:=a^2*b^5 - a^2*b^3*c - a*b^6 + a*b^4*c + b^8 + b^7 -  b^5*c + b^4*d + b^4*e + 2*b^4 + b
f27:=-a^2*b^5 + a^2*b^3*c + a*b^6 - a*b^4*c - b^8 - b^7 + b^5*c - 2*b^4 - b^3*c^2 + b^3*e -




I:=ideal<P|f1,f2,f3,f4,f5,f6,f7,f8,f9,f10,f11,f12,f13,f14,f15,f16,f17,f18,f19,f20,f21,f22,f2
```

```
G:=GroebnerBasis(I);G;#G;
```

is readable, with last line an image of the input polynomial:

```
e^2 - d + c^2 - b^8 - b^7 + b^6*a - b^5*a^2 + b^4*a^3 - b^4 - b^3*a^4 +
    b^2*a^6 + b^2*a^5 - b^2*a^2 - a^8,
e*d + e + d + b^8 + b^7 - b^6*a + b^5*a^2 - b^4*a^3 + 2*b^4 + b^3*a^4 +
    b^3*a + b^3 - b^2*a^5 + b^2*a^2 + b*a^7 + b*a^6 + a^8 + 1,
e*c - c^2*a^2 - b^6 - b^5 + b^4*a^2 + b^4*a - b^2,
e*b - c^2*a - a,
e*a + d*a + b^5 + b^4*a + b^4 + b + a,
d^2 - b^8 - b^7 + b^6*a - b^5*a^2 + b^4*a^3 - 2*b^4 - b^3*a^4 - b^3 +
    b^2*a^5 + b^2*a - b*a^6 + a^7 - 1,
d*c + c^2*a^2 + c + b^6 + b^5 - b^4*a + b^3*a^3 + b^3*a^2 + b^2 + a^2,
d*b + d*a + b^5 + b^4*a + b^4 + b + a,
d*a^2 - c^2*a^2 - b^6 - b^5 + b^4*a^2 + b^4*a - b^2,
c^3 + c^2*a^2 + c + b^6 + b^5*a + b^5 + b^2 + b*a + a^2,
c^2*a^3 + c*a^5 + b^7 + b^6*a + b^6 + b^3 + b^2*a + b*a^2 + a^3,
c*b - a^3,
c*a^6 + b^8 + b^7*a + b^7 + b^4 + b^3*a + b^2*a^2 + b*a^3 + a^8,
b^9 + b^8*a + b^8 + b^5 + b^4*a + b^3*a^2 + b^2*a^3 + b*a^8 + a^9
```

meaning it is relatively easy to read off that $c = a^3/b$, $d = -(b^5 + b^4 * a + b^4 + b + a)/(b + a)$ and $e = ((a^3/b)^2 + 1)/b$.