Singh and Swanson example

1 Example 2.3

The basic example, 2.3, from the 2009 paper of Singh and Swanson is

$$R := \mathbf{F}_p[u,v,x,y,z]/\langle u^2x^p + 2uvy^p + v^2z^p\rangle.$$

Make this an integral extension problem by using the variable $t := ux^p/v$ to get

$$R_1 := \mathbf{F}_p[t,x,y,z]/\langle t^2 + 2ty^p + x^pz^p\rangle.$$

But noticing that all the coefficients are pth powers with p the characteristic, it is possible to use the variable $s := t^{1/p}$ to get

$$R_2 := \mathbf{F}_p[s,x,y,z]/\langle s^2 + 2sy + xz\rangle,$$

which is integrally closed.

From $vs^p - ux^p = 0$, use the variable $r := vs/x$ to make this into an integral extension:

$$R_3 := \mathbf{F}_p[r,u,v]/\langle r^p - uv^{p-1}\rangle.$$

The integral closure of this is generated over $\mathbf{F}_p[u,v]$ by the elements $r^i/v^{i-1} = (u^iv^{p-i})^{1/p}$ for $0 \le i \le p$.

This gives a presentation of the integral closure as

$$\mathbf{F}_p[r,s,u,v,x,y,z]/\langle s^2 + 2sy + xz, rx - sv, r^p - uv^{p-1}\rangle$$

succinctly describing s integral over $\mathbf{F}_p[x,y,z]$, r integral over $\mathbf{F}_p[u,v]$, and the relation between them.

This is in line with the theory in the paper; but look at what is actually produced by the various implementations absent this pre-processing. Can anything be gained by considering the explicit fractions produced by any of these?

2 Macaulay2

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : p=5;

i2 : S=ZZ/p[u,v,x,y,z];

i3 : P=ideal(u^2*x^p+2*u*v*y^p+v^2*z^p);

o3 : Ideal of S

i4 : R=S/P;

i5 : time toString(icFracP(R))
     -- used 0.139869 seconds

o5 = {1, (-u*y^5-v*z^5)/(u*y^4-2*u*x*y^2*z+u*x^2*z^2),
     (-u*x*y^3-2*u*x^2*y*z+2*v*z^4)/(y^4-2*x*y^2*z+x^2*z^2),
     (-u*x^2*y^2-u*x^3*z-2*v*y*z^3)/(y^4-2*x*y^2*z+x^2*z^2),
     (-u*x^3*y+2*v*y^2*z^2+2*v*x*z^3)/(y^4-2*x*y^2*z+x^2*z^2),
     (-u*x^4-2*v*y^3*z+v*x*y*z^2)/(y^4-2*x*y^2*z+x^2*z^2)}

i6 : time toString(icFractions(R))
     -- used 125.845 seconds

o6 = {(-2*u*x^2*y^2-2*u*x^3*z+v*y*z^3)/(y^4-2*x*y^2*z+x^2*z^2),
     (u*x^3*y-2*v*y^2*z^2-2*v*x*z^3)/(y^4-2*x*y^2*z+x^2*z^2),
     (-2*u*x*y^4-u*x^2*y^2*z-2*u*x^3*z^2)/(u*x*y^3+2*u*x^2*y*z-2*v*z^4),
     (u*x^4+2*v*y^3*z-v*x*y*z^2)/(y^4-2*x*y^2*z+x^2*z^2),
     (2*u*x*y^3-u*x^2*y*z+v*z^4)/(y^4-2*x*y^2*z+x^2*z^2), u, v, x, y, z}
```

3 Singular

```
                    SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-2
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   Oct 2010
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
LIB "normal.lib";
int p=5;
ring r=p,(u,v,x,y,z),dp;
ideal i=u^2*x^p+2*u*v*y^p+v^2*z^p;
list nor=normalP(i); nor;
[1]:
   [1]:
      _[1]=uxy4z-ux2y2z2+2ux3z3+2vyz5
      _[2]=u2xy4-u2x2y2z+2u2x3z2+2uvyz4
      _[3]=u2x2y3-2u2x3yz-2uvy2z3+uvxz4
      _[4]=u2x3y2+2u2x4z+2uvy3z2+uvxyz3
      _[5]=u2x4y-2uvy4z+2uvxy2z2+uvx2z3
      _[6]=uy5+vz5

 def R=nor[1][1];
 setring R;
 norid;
norid[1]=T(2)*x-T(3)*y-T(4)*z
norid[2]=T(3)*x-T(4)*y-T(5)*z
norid[3]=T(1)*u-T(2)*z
norid[4]=-2*T(4)*x+2*T(5)*y+v*z
norid[5]=T(1)*v-2*T(5)*x+v*y
norid[6]=-T(2)*y-T(3)*z+u*x
norid[7]=-2*T(1)*v*z^5+T(4)*x*y*z^4-T(5)*y^2*z^4-T(5)*x*z^5
norid[8]=T(1)^2+T(1)*y-x*z
norid[9]=T(1)*T(2)-T(3)*z
norid[10]=T(2)^2-T(3)*u
norid[11]=T(1)*T(3)-T(4)*z
norid[12]=T(2)*T(3)-T(4)*u
norid[13]=T(3)^2-T(5)*u
norid[14]=T(1)*T(4)-T(5)*z
norid[15]=T(2)*T(4)-T(5)*u
norid[16]=T(3)*T(4)+2*u*v
norid[17]=T(4)^2+2*T(2)*v
norid[18]=T(1)*T(5)+2*v*z
norid[19]=T(2)*T(5)+2*u*v
norid[20]=T(3)*T(5)+2*T(2)*v
norid[21]=T(4)*T(5)+2*T(3)*v
norid[22]=T(5)^2+2*T(4)*v
```

```
norid[23]=u^2*x^5+2*u*v*y^5+v^2*z^5
 ideal s=std(norid);s;
s[1]=u^2*x^5+2*u*v*y^5+v^2*z^5
s[2]=T(5)*y^4-2*T(5)*x*y^2*z+T(5)*x^2*z^2-u*x^4-2*v*y^3*z+v*x*y*z^2
s[3]=T(5)*u*x*y^3+2*T(5)*u*x^2*y*z-2*T(5)*v*z^4+2*u*v*y^4+u*v*x*y^2*z+2*u*v*x^2*z^2
s[4]=T(5)*u*x^2*y^2+T(5)*u*x^3*z+2*T(5)*v*y*z^3+2*u*v*x*y^3-u*v*x^2*y*z+v^2*z^4
s[5]=T(5)*u*x^3*y-2*T(5)*v*y^2*z^2-2*T(5)*v*x*z^3+2*u*v*x^2*y^2+2*u*v*x^3*z-v^2*y*z^3
s[6]=T(5)*u*x^4+2*T(5)*v*y^3*z-T(5)*v*x*y*z^2+2*u*v*x^3*y+v^2*y^2*z^2+v^2*x*z^3
s[7]=T(4)*x-T(5)*y+2*v*z
s[8]=T(4)*y^3+2*T(4)*x*y*z+T(5)*y^2*z+T(5)*x*z^2-u*x^3
s[9]=T(4)*v*z^3-2*T(5)*u*x*y^2-2*T(5)*u*x^2*z+u*v*y^3+2*u*v*x*y*z
s[10]=T(4)*v*y*z^2+2*T(5)*u*x^2*y+T(5)*v*z^3-u*v*x*y^2-u*v*x^2*z
s[11]=T(4)*v*y^2*z+T(4)*v*x*z^2-2*T(5)*u*x^3+T(5)*v*y*z^2+u*v*x^2*y
s[12]=T(3)*x-T(4)*y-T(5)*z
s[13]=T(3)*y^2+T(3)*x*z+T(4)*y*z-u*x^2
s[14]=T(3)*v*z^2+2*T(5)*u*x*y-u*v*y^2-u*v*x*z
s[15]=T(3)*v*y*z+T(4)*v*z^2-2*T(5)*u*x^2+u*v*x*y
s[16]=T(2)*y+T(3)*z-u*x
s[17]=T(2)*x-T(3)*y-T(4)*z
s[18]=T(2)*v*z-2*T(5)*u*x+u*v*y
s[19]=T(1)*v-2*T(5)*x+v*y
s[20]=T(1)*u-T(2)*z
s[21]=T(5)^2+2*T(4)*v
s[22]=T(4)*T(5)+2*T(3)*v
s[23]=T(3)*T(5)+2*T(2)*v
s[24]=T(2)*T(5)+2*u*v
s[25]=T(1)*T(5)+2*v*z
s[26]=T(4)^2+2*T(2)*v
s[27]=T(3)*T(4)+2*u*v
s[28]=T(2)*T(4)-T(5)*u
s[29]=T(1)*T(4)-T(5)*z
s[30]=T(3)^2-T(5)*u
s[31]=T(2)*T(3)-T(4)*u
s[32]=T(1)*T(3)-T(4)*z
s[33]=T(2)^2-T(3)*u
s[34]=T(1)*T(2)-T(3)*z
s[35]=T(1)^2+T(1)*y-x*z
```

```
 int p=5;
 ring r=p,(u,v,x,y,z),dp;
 ideal i=u^2*x^p+2*u*v*y^p+v^2*z^p;
 list nor=normal(i); nor;
//   characteristic : 5
//   number of vars : 10
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3) T(4) T(5)
//        block   2 : ordering dp
//                  : names    u v x y z
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=uv2xy3+2uv2x2yz-2v3z4
      _[2]=uv2x2y2+uv2x3z+2v3yz3
      _[3]=uv2x3y-2v3y2z2-2v3xz3
      _[4]=uvx5+2v2xy3z-v2x2yz2
      _[5]=uv2x4+2v3y3z-v3xyz2
      _[6]=v2y4-2v2xy2z+v2x2z2
 def R=nor[1][1];
 setring R;
 norid;
norid[1]=T(1)*x-T(2)*y-T(3)*z
norid[2]=T(2)*x-T(3)*y-T(5)*z
norid[3]=T(4)*v-T(5)*x
norid[4]=-2*T(3)*x+2*T(5)*y+v*z
norid[5]=T(1)*z-2*T(4)*u+u*y
norid[6]=-T(1)*y-T(2)*z+u*x
norid[7]=2*T(1)*y^2-T(2)*y*z+2*T(3)*z^2+T(4)*u*x
norid[8]=-2*T(1)*v*z^2+T(3)*u*x*y-T(5)*u*y^2-T(5)*u*x*z
norid[9]=2*T(1)*v*y*z+2*T(2)*v*z^2+T(3)*u*x^2-T(5)*u*x*y
norid[10]=2*T(1)*v*y^2-T(2)*v*y*z+2*T(3)*v*z^2+T(5)*u*x^2
norid[11]=T(1)^2-T(2)*u
norid[12]=T(1)*T(2)-T(3)*u
norid[13]=T(2)^2-T(5)*u
norid[14]=T(1)*T(3)-T(5)*u
norid[15]=T(2)*T(3)+2*u*v
norid[16]=T(3)^2+2*T(1)*v
norid[17]=T(1)*T(4)+2*u*x
norid[18]=T(2)*T(4)+2*T(1)*x
norid[19]=T(3)*T(4)+2*T(2)*x
norid[20]=T(4)^2+2*T(4)*y+x*z
norid[21]=T(1)*T(5)+2*u*v
norid[22]=T(2)*T(5)+2*T(1)*v
norid[23]=T(3)*T(5)+2*T(2)*v
norid[24]=T(4)*T(5)+2*T(5)*y+v*z
```

```
norid[25]=T(5)^2+2*T(3)*v
norid[26]=u^2*x^5+2*u*v*y^5+v^2*z^5
 ideal s=std(norid);s;
s[1]=u^2*x^5+2*u*v*y^5+v^2*z^5
s[2]=T(5)*y^4-2*T(5)*x*y^2*z+T(5)*x^2*z^2-u*x^4-2*v*y^3*z+v*x*y*z^2
s[3]=T(5)*u*x*y^3+2*T(5)*u*x^2*y*z-2*T(5)*v*z^4+2*u*v*y^4+u*v*x*y^2*z+2*u*v*x^2*z^2
s[4]=T(5)*u*x^2*y^2+T(5)*u*x^3*z+2*T(5)*v*y*z^3+2*u*v*x*y^3-u*v*x^2*y*z+v^2*z^4
s[5]=T(5)*u*x^3*y-2*T(5)*v*y^2*z^2-2*T(5)*v*x*z^3+2*u*v*x^2*y^2+2*u*v*x^3*z-v^2*y*z^3
s[6]=T(5)*u*x^4+2*T(5)*v*y^3*z-T(5)*v*x*y*z^2+2*u*v*x^3*y+v^2*y^2*z^2+v^2*x*z^3
s[7]=T(4)*v-T(5)*x
s[8]=T(4)*u*y^3+2*T(4)*u*x*y*z-2*T(5)*z^4+2*u*y^4+u*x*y^2*z+2*u*x^2*z^2
s[9]=T(4)*u*x*y^2+T(4)*u*x^2*z+2*T(5)*y*z^3+2*u*x*y^3-u*x^2*y*z+v*z^4
s[10]=T(4)*u*x^2*y-2*T(5)*y^2*z^2-2*T(5)*x*z^3+2*u*x^2*y^2+2*u*x^3*z-v*y*z^3
s[11]=T(4)*u*x^3+2*T(5)*y^3*z-T(5)*x*y*z^2+2*u*x^3*y+v*y^2*z^2+v*x*z^3
s[12]=T(3)*x-T(5)*y+2*v*z
s[13]=T(3)*z^3-2*T(4)*u*y^2-2*T(4)*u*x*z+u*y^3+2*u*x*y*z
s[14]=T(3)*y*z^2+2*T(4)*u*x*y+T(5)*z^3-u*x*y^2-u*x^2*z
s[15]=T(3)*y^2*z+T(3)*x*z^2-2*T(4)*u*x^2+T(5)*y*z^2+u*x^2*y
s[16]=T(3)*y^3+2*T(3)*x*y*z+T(5)*y^2*z+T(5)*x*z^2-u*x^3
s[17]=T(2)*x-T(3)*y-T(5)*z
s[18]=T(2)*z^2+2*T(4)*u*y-u*y^2-u*x*z
s[19]=T(2)*y*z+T(3)*z^2-2*T(4)*u*x+u*x*y
s[20]=T(2)*y^2+T(2)*x*z+T(3)*y*z-u*x^2
s[21]=T(1)*z-2*T(4)*u+u*y
s[22]=T(1)*y+T(2)*z-u*x
s[23]=T(1)*x-T(2)*y-T(3)*z
s[24]=T(5)^2+2*T(3)*v
s[25]=T(4)*T(5)+2*T(5)*y+v*z
s[26]=T(3)*T(5)+2*T(2)*v
s[27]=T(2)*T(5)+2*T(1)*v
s[28]=T(1)*T(5)+2*u*v
s[29]=T(4)^2+2*T(4)*y+x*z
s[30]=T(3)*T(4)+2*T(2)*x
s[31]=T(2)*T(4)+2*T(1)*x
s[32]=T(1)*T(4)+2*u*x
s[33]=T(3)^2+2*T(1)*v
s[34]=T(2)*T(3)+2*u*v
s[35]=T(1)*T(3)-T(5)*u
s[36]=T(2)^2-T(5)*u
s[37]=T(1)*T(2)-T(3)*u
s[38]=T(1)^2-T(2)*u
```

4 Magma

```
Magma V2.15-9     Sat Jan 15 2011 21:37:53 on dell-desktop [Seed = 1011030395]
Type ? for help.  Type <Ctrl>-D to quit.
> q:=5;
> S<u,v,x,y,z>:=PolynomialRing(GF(q),5);
> P:=ideal<S|u^2*x^q+2*u*v*y^q+v^2*z^q>;
> J:=Normalisation(P);
> J1:=J[1][1];
> G1:=GroebnerBasis(J1);G1;
[
    $.1^2 + 4*$.8*$.10 + $.9^2,
    $.1*$.2 + 3*$.3*$.10 + $.6*$.8,
    $.1*$.3 + 2*$.3*$.9 + 2*$.4*$.10,
    $.1*$.4 + 2*$.4*$.9 + 3*$.5*$.10,
    $.1*$.5 + 2*$.5*$.9 + 4*$.7*$.10,
    $.1*$.6 + 3*$.2*$.10 + 2*$.6*$.9,
    $.1*$.7 + 4*$.5*$.8 + 3*$.7*$.9,
    $.2^2 + 3*$.3*$.6,
    $.2*$.3 + $.4*$.6,
    $.2*$.4 + 4*$.5*$.6,
    $.2*$.5 + 2*$.6*$.7,
    $.2*$.7 + 4*$.4^2,
    $.2*$.8 + $.3*$.9 + 3*$.4*$.10,
    $.2*$.9 + 4*$.3*$.10 + 2*$.6*$.8,
    $.3^2 + 3*$.5*$.6,
    $.3*$.4 + $.6*$.7,
    $.3*$.5 + $.4^2,
    $.3*$.7 + 2*$.4*$.5,
    $.3*$.8 + 2*$.4*$.9 + 4*$.5*$.10,
    $.3*$.9^2 + $.4*$.9*$.10 + $.5*$.10^2 + 3*$.6*$.8^2,
    $.4^3 + 4*$.5*$.6*$.7,
    $.4^2*$.5 + 2*$.6*$.7^2,
    $.4^2*$.9 + 2*$.4*$.5*$.10 + 2*$.6*$.7*$.8,
    $.4^2*$.10 + 2*$.5*$.6*$.8 + 3*$.6*$.7*$.9,
    $.4*$.5^3 + $.6*$.7^3,
    $.4*$.5^2*$.9 + 2*$.5^3*$.10 + $.6*$.7^2*$.8,
    $.4*$.5^2*$.10 + $.5*$.6*$.7*$.8 + 4*$.6*$.7^2*$.9,
    $.4*$.5*$.9^2 + 4*$.5^2*$.9*$.10 + 2*$.5*$.7*$.10^2 + $.6*$.7*$.8^2,
    $.4*$.5*$.9*$.10 + 2*$.5^2*$.10^2 + $.5*$.6*$.8^2 + 4*$.6*$.7*$.8*$.9,
    $.4*$.5*$.10^2 + 4*$.5*$.6*$.8*$.9 + $.6*$.7*$.8*$.10 + $.6*$.7*$.9^2,
    $.4*$.7 + 3*$.5^2,
    $.4*$.8 + 3*$.5*$.9 + 3*$.7*$.10,
    $.4*$.9^3 + 2*$.5*$.8*$.10^2 + $.5*$.9^2*$.10 + $.6*$.8^3 +
        4*$.7*$.9*$.10^2,
    $.5^5 + 3*$.6*$.7^4,
```

```
    $.5^4*$.9 + $.5^3*$.7*$.10 + 3*$.6*$.7^3*$.8,
    $.5^4*$.10 + 3*$.5*$.6*$.7^2*$.8 + 2*$.6*$.7^3*$.9,
    $.5^3*$.9^2 + 2*$.5^2*$.7*$.9*$.10 + $.5*$.7^2*$.10^2 + 3*$.6*$.7^2*$.8^2,
    $.5^3*$.9*$.10 + $.5^2*$.7*$.10^2 + 3*$.5*$.6*$.7*$.8^2 +
        2*$.6*$.7^2*$.8*$.9,
    $.5^3*$.10^2 + 2*$.5*$.6*$.7*$.8*$.9 + 3*$.6*$.7^2*$.8*$.10 +
        3*$.6*$.7^2*$.9^2,
    $.5^2*$.8 + 4*$.5*$.7*$.9 + 4*$.7^2*$.10,
    $.5^2*$.9^3 + $.5*$.7*$.8*$.10^2 + 3*$.5*$.7*$.9^2*$.10 + 3*$.6*$.7*$.8^3 +
        2*$.7^2*$.9*$.10^2,
    $.5^2*$.9^2*$.10 + 3*$.5*$.6*$.8^3 + 2*$.5*$.7*$.9*$.10^2 +
        2*$.6*$.7*$.8^2*$.9 + $.7^2*$.10^3,
    $.5^2*$.9*$.10^2 + 2*$.5*$.6*$.8^2*$.9 + $.5*$.7*$.10^3 +
        3*$.6*$.7*$.8^2*$.10 + 3*$.6*$.7*$.8*$.9^2,
    $.5^2*$.10^3 + 3*$.5*$.6*$.8^2*$.10 + 3*$.5*$.6*$.8*$.9^2 +
        4*$.6*$.7*$.8*$.9*$.10 + 2*$.6*$.7*$.9^3,
    $.5*$.6*$.8^4 + 4*$.5*$.7*$.8*$.9*$.10^2 + 2*$.5*$.7*$.9^3*$.10 +
        4*$.6*$.7*$.8^3*$.9 + 2*$.7^2*$.8*$.10^3 + 2*$.7^2*$.9^2*$.10^2,
    $.5*$.6*$.8^3*$.9 + 3*$.5*$.7*$.8*$.10^3 + 3*$.5*$.7*$.9^2*$.10^2 +
        4*$.6*$.7*$.8^3*$.10 + 4*$.6*$.7*$.8^2*$.9^2 + 3*$.7^2*$.9*$.10^3,
    $.5*$.6*$.8^3*$.10 + $.5*$.6*$.8^2*$.9^2 + 2*$.5*$.7*$.9*$.10^3 +
        3*$.6*$.7*$.8^2*$.9*$.10 + 4*$.6*$.7*$.8*$.9^3 + 2*$.7^2*$.10^4,
    $.5*$.6*$.8^2*$.9^3 + 2*$.5*$.7*$.8*$.10^4 + 4*$.5*$.7*$.9^2*$.10^3 +
        $.6*$.7*$.8^3*$.10^2 + 4*$.6*$.7*$.8^2*$.9^2*$.10 + 4*$.6*$.7*$.8*$.9^4
        + 4*$.7^2*$.9*$.10^4,
    $.5*$.6*$.8^2*$.9*$.10 + 3*$.5*$.6*$.8*$.9^3 + 4*$.5*$.7*$.10^4 +
        2*$.6*$.7*$.8^2*$.10^2 + $.6*$.7*$.8*$.9^2*$.10 + 2*$.6*$.7*$.9^4,
    $.5*$.6*$.9^5 + $.5*$.7*$.10^5 + 3*$.6^2*$.8^4*$.9 + 3*$.6*$.7*$.8^2*$.10^3
        + $.6*$.7*$.8*$.9^2*$.10^2 + 4*$.6*$.7*$.9^4*$.10,
    $.5*$.8^2*$.10^2 + 3*$.5*$.8*$.9^2*$.10 + $.5*$.9^4 + 3*$.6*$.8^4 +
        2*$.7*$.8*$.9*$.10^2 + $.7*$.9^3*$.10,
    $.6^2*$.8^5 + 2*$.6*$.7*$.9^5 + $.7^2*$.10^5
]
```

5 Reducibilty

An alternative approach would be to work in a larger ring

$$\mathbf{F}_p[a, b, u, v, x, y, z]/\langle u^2x^p + 2uvy^p + v^2z^p, a^p - u, b^p - v\rangle.$$

There

$$u^2x^p + 2uvy^p + v^2z^p = (a^2x + 2aby + b^2z)^p.$$

Then ax/b is integral over $\mathbf{F}_p[x, y, z]$, and av/b is integral over $\mathbf{F}_p[u, v]$ as above.

Either approach highlights the fact that this is really a disguised reducible problem with repeated factors. [The theory in the paper seemed to be meant to apply to exactly the opposite type of problem.]