1 Example 3.7.3 from the SINGULAR book

Example 3.7.3 from the SINGULAR book is written as

```
zy2-zx3-x6,
```

with a wp(6,3,2) monomial ordering suggested.

In my opinion this should have been written as $x^6+x^3z-y^2z$ so as to view it as an integral extension problem, extending $\mathbf{F}[y,z]$ to $\mathbf{F}[x;y,z]/\langle x^6+x^3z-y^2z\rangle$.

The monomial ordering probably only makes sense in that all three terms would then have the same weight, making it a weighted homogeneous example. There is no mathematics in the book to support such a view, so this is just conjecture.

2 A slight variant

Let's try the example

$$A := \mathbf{F}[x,y,z]/\langle x^6 + x^3 z - y^3 z^2 \rangle,$$

which has the same flavor but a slightly more interesting integral closure.

This has a natural weight function $W_A := \begin{pmatrix} 5 & 6 & 6 \\ 3 & 6 & 0 \end{pmatrix}$, extending the grevlex monomial ordering $W_P := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ on $P := \mathbf{F}[y,z]$, the implicit Noether normalization of A.

This should have an integral closure $C(P, Q(A))$ with P-module basis:

$$f_{10,6} := x^2,\ f_{9,9} := x^3/z,\ f_{8,6} := (x^4 + xz)/(yz),$$

$$f_{7,3} := (x^5 + x^2 z)/(y^2 z),\ f_{5,3} := x,\ f_{0,0} := 1,$$

and hence 15 non-trivial quadratic relations defining the P-algebra multiplication, as below in my qth-power outputs in either Macaulay2 or Magma.

To be preachy about it, a Gröbner basis should actually contain information. That is, it should tell how to reduce a leading term to terms lower in the ordering. Here, the idea would be to reduce products of the non-trivial P-module basis elements to P-linear combinations of P-module basis elements. But to make this happen, it is necessary at the very least to choose an appropriate monomial ordering. [And I want information such as $NormalForm(x^5 + x^2 z, \overline{I}) = f_{7,3} y^2 z$, telling me that $(x^5 + x^2 z)/(y^2 z)$ is an element of the integral closure here.]

3 Singular

```
                    SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-3
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   March 2011
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "normal.lib";
> intmat A[3][3]=5,6,6,3,6,0,1,0,0;
> ring r=0,(x,y,z),M(A);
> int time=timer;
> ideal i=x^6+x^3*z-y^3*z^2;
> list nor=normal(i);nor;
//   characteristic : 0
//   number of vars : 6
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3)
//        block   2 : ordering M
//                  : names    x y z
//                  : weights  5 6 6
//                  : weights  3 6 0
//                  : weights  1 0 0
//        block   3 : ordering C
      _[1]=y3z
      _[2]=xy2z
      _[3]=x2yz
      _[4]=x3
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=x^6-y^3*z^2+x^3*z
s[2]=T(3)*x-y*z
s[3]=T(3)*y^2*z-x^5-x^2*z
s[4]=T(2)*x-T(3)*y
s[5]=T(2)*y*z-x^4-x*z
s[6]=T(1)*x-T(2)*y
s[7]=T(1)*z-x^3-z
s[8]=T(3)^2-T(2)*z
s[9]=T(2)*T(3)-x^3-z
s[10]=T(1)*T(3)-T(3)-x^2*y
s[11]=T(2)^2-T(3)-x^2*y
s[12]=T(1)*T(2)-T(2)-x*y^2
s[13]=T(1)^2-T(1)-y^3
```

A strict affine A-algebra presentation is not the worst thing that could happen to this example (though I believe the strict affine P-algebra one to be better), except that this one ignores the weighted monomial ordering that should have been induced by A.

And the default product monomial ordering, grevlex-over-input does nothing to help in deciding membership in the integral closure. That would undoubtedly have to be done by looking at the fractions produced, requiring elements of $Q(A)$ to be rewritten with denominator x^3. Not all elements of $Q(A)$ can be written that way, and proably not all elements of $C(A, Q(A))$ can be written uniquely with elements in a Noether normalization including x.

```
>  ring r=17,(x,y,z),M(A);
> ideal i=x6+x3z-y3z2;
> int time=timer;
> list nor=normalP(i,"withRing");nor;
//   characteristic : 17
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    T(2) T(3)
//        block   2 : ordering dp
//                  : names    x y z
//        block   3 : ordering C
      _[1]=x2yz
      _[2]=xy2z
      _[3]=y3z
      _[4]=x3
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=x^6-y^3*z^2+x^3*z
s[2]=T(3)*z-x^3-z
s[3]=T(3)*x^3-y^3*z
s[4]=T(2)*y-T(3)*x
s[5]=T(2)*x^2-y^2*z
s[6]=T(3)^2-T(3)-y^3
s[7]=T(2)*T(3)-T(2)-x*y^2
s[8]=T(2)^2*x-x^3*y-y*z
s[9]=T(2)^3-y^3*z-x^3-z
```

This suffers from the problems mentioned above with respect to `normal`; but it no longer even has an A-module generating set, missing what was called T[1] before it was removed. This makes it even harder to find out that elements such as T[1] actually are elements of C(A,Q(A)).

4 Macaulay2

```
i2 : R0=QQ[x,y,z,Weights=>{5,6,6},Weights=>{3,6,0},Weights=>{1,0,0}];

i3 : I0=ideal(x^6+x^3*z-y^3*z^2);

i4 : GQ=transpose gens gb presentation integralClosure(R0/I0)


o4 = {-6} | x6-y3z2+x3z                   |
     {-3} | w_(1,1)x2-y2z                 |
     {-4} | w_(1,1)yz-x4-xz               |
     {-4} | w_(1,1)^2x-x3y-yz             |
     {-4} | w_(1,1)^2y-w_(1,1)x-x2y2      |
     {-4} | w_(1,1)^3-y3z-x3-z            |
     {-2} | w_(2,0)x-w_(1,1)y             |
     {-3} | w_(2,0)z-x3-z                 |
     {-3} | w_(2,0)w_(1,1)-w_(1,1)-xy2    |
     {-3} | w_(2,0)^2-w_(2,0)-y3          |
```

Clearly (well only if you believe in reversing normal form computations) $w_{1,1} = (x^4 + xz)/(yz)$ and $w_{2,0} = (x^3 + z)/z$. Is it easy to see that $(x^5 + x^2z)/(y^2z)$ is in the integral closure? For me to do this from this presentation, I would take the relation $w_{1,1}^2x - x^3y - yz$ to see that $w_{1,1}^2 - x^2y = yz/x$ should be there, and then rewrite it as $(x^5 + x^2z)/(y^2z)$. The point of this is that a generic affine A-algebra presentation is not conducive to membership, which should be done by computing *normal forms* from the given presentation, not by essentially producing a P-module basis after the fact to get a better presentation.

[Another way of thinking of this is that $w_{1,1}^2$ and x^2y are both standard monomials relative to the ideal of relations, so already in normal form. It would be nice if any **F**-linear combination of them had normal form with leading monomial either $w_{1,1}^2$ or x^2y; but clearly $w_{1,1}^2 - x^2y = yz/x$ doesn't.]

```
Macaulay2, version 1.4

i1 : A=ZZ/17[x,y,z]/(x^6+x^3*z-y^3*z^2);

i2 : icp=icFracP A;

i3 : toString icp

o3 = {x,
      1,
     (x^3-8*z)/(z),
     (y^2*z)/(x^2),
     (y*z)/(x)}
```

Notice that icFracP does produce $(x^5 + x^2z)/(y^2z)$, but as yz/x (just as $(x^4 + xz)/(yz) = y^2z/x^2$). This is because it allows Macaulay2 to micromanage the writing of fractions. This can be avoided by not using quotient rings in the code so that this point is moot. Note also that icFracP doesn't produce a P-module generating set, since x^2 is missing; and doesn't even try for a presentation, meaning we have no clue what the mindset of the implementers was in this regard. (Well, we do in the sense that they must be satisfied with an answer of the form $\sum_i Af_i$ for f_i in the output list of fractions.)

5 Magma

```
t:=Cputime();
Q:=Rationals();
P<x,y,z>:=PolynomialRing(Q,3,
          "weight",
          [5,6,6,
           3,6,0,
           1,0,0]);
f:=x^6+x^3*z-y^3*z^2;
I:=ideal<P|f>;
J:=Normalisation(I:FFMin:=true);
"J=",J;
Cputime(t);
G:=GroebnerBasis(J[1][1]);G;#G;
Cputime(t);
```

The output (taking around 300 seconds to produce) from describing J takes over $14,000$ lines, so is omitted here. The GroebnerBasis output, taking another 300 seconds) is

```
    $.1 + 30*$.3 - 108*$.4*$.8 - 2/81*$.8^4 - 1/162*$.8,
    $.2 - 48*$.3*$.8 + 432*$.4*$.8^2 + 8/81*$.8^5 - 1/81*$.8^2,
    $.3^2 + 11/972*$.3*$.8 - 4/27*$.4*$.8^5 - 5/27*$.4*$.8^2 - 1/59049*$.8^11 -
        1/78732*$.8^8 - 11/944784*$.8^5,
    $.3*$.4 + 1/2187*$.3 - 7/486*$.4*$.8^4 - 5/486*$.4*$.8 - 1/531441*$.8^10 -
        1/2125764*$.8^7 - 1/2125764*$.8^4,
    $.3*$.5 - 5/1944*$.5*$.8 - 5/34992*$.6*$.8^3 - 1/39366*$.7*$.8^8,
    $.3*$.6 - 4/27*$.5*$.8^2 - 1/81*$.6*$.8^4 - 5/972*$.6*$.8 - 4/2187*$.7*$.8^9
        - 1/729*$.7*$.8^6 + 5/8748*$.7*$.8^3,
    $.3*$.7 - 2/3*$.5 - 1/108*$.6*$.8^2 - 1/243*$.7*$.8^4,
    $.3*$.8^3 + $.3 - 9*$.4*$.8^4 - 45/4*$.4*$.8 - 1/486*$.8^7 - 1/972*$.8^4,
    $.3*$.9 - 1/243*$.7^3*$.8^4 - 5/972*$.7^3*$.8 - 5/486*$.8^4*$.9 -
        5/972*$.8*$.9,
    $.4^2 - 1/729*$.4*$.8^3 - 1/2187*$.4 - 1/4782969*$.8^9,
    $.4*$.5 - 1/4374*$.5 - 1/78732*$.6*$.8^2 - 1/354294*$.7*$.8^7,
    $.4*$.6 - 1/81*$.5*$.8 - 5/4374*$.6*$.8^3 - 1/2187*$.6 - 4/19683*$.7*$.8^8 -
        2/19683*$.7*$.8^5 + 1/19683*$.7*$.8^2,
    $.4*$.7 + 2/9*$.5*$.8^2 - 1/243*$.6*$.8 + 1/2187*$.7*$.8^3,
    $.4*$.9 - 1/2187*$.7^3*$.8^3 - 1/2187*$.7^3 - 2/2187*$.8^3*$.9 - 1/2187*$.9,
    $.5^2 - 1/162*$.5*$.7*$.8 - 1/26244*$.7^2*$.8^5,
    $.5*$.6 - 2/9*$.5*$.7*$.8^2 - 1/162*$.6*$.7*$.8 - 2/729*$.7^2*$.8^6 +
        1/1458*$.7^2*$.8^3,
    $.5*$.7^2 - 1/162*$.8^4*$.9,
    $.5*$.8^3 + 1/4*$.5 - 1/72*$.6*$.8^2 + 1/324*$.7*$.8^4,
    $.5*$.9 - 1/162*$.7^4*$.8 - 1/162*$.7*$.8*$.9,
```

```
$.6^3 - 160/729*$.7^3*$.8^9 - 104/729*$.7^3*$.8^6 - 2/81*$.7^3*$.8^3 -
    1/729*$.7^3 - 64/729*$.8^12*$.9 - 256/729*$.8^9*$.9 - 40/243*$.8^6*$.9 -
    19/729*$.8^3*$.9 - 1/729*$.9,
$.6^2*$.7 - 16/81*$.7^3*$.8^7 - 4/27*$.7^3*$.8^4 - 1/81*$.7^3*$.8 -
    32/81*$.8^7*$.9 - 4/27*$.8^4*$.9 - 1/81*$.8*$.9,
$.6^2*$.8 - 8/9*$.6*$.7*$.8^3 - 1/9*$.6*$.7 - 16/81*$.7^2*$.8^8 +
    4/81*$.7^2*$.8^5 + 1/81*$.7^2*$.8^2,
$.6*$.7^2 - 2/9*$.7^3*$.8^2 - 4/9*$.8^5*$.9 - 1/9*$.8^2*$.9,
$.6*$.8*$.9 - 4/9*$.7^4*$.8^3 - 1/9*$.7^4 - 2/3*$.7*$.8^3*$.9 - 1/9*$.7*$.9,
$.7^6 + $.7^3*$.9 - $.8^3*$.9^2
```

From this, it is possible to see (using S as the name of the ring and $S.7 = x$, $S.8 = y$, $S.9 = z$) that:

$$S.6 * y * z - 4/9 * x^4 * y^3 - 1/9 * x^4 - 2/3 * x * y^3 * z - 1/9 * x * z = 0,$$

$$S.5 * z - 1/162 * x^4 * y - 1/162 * x * y * z = 0,$$

$$S.4 * z - 1/2187 * x^3 * y^3 - 1/2187 * x^3 - 2/2187 * y^3 * z - 1/2187 * z = 0,$$

$$S.3 * z - 1/243 * x^3 * y^4 - 5/972 * x^3 * y - 5/486 * y^4 * z - 5/972 * y * z = 0,$$

$$S.2 - 48 * S.3 * y + 432 * S.4 * y^2 + 8/81 * y^5 - 1/81 * y^2 = 0,$$

$$S.1 + 30 * S.3 - 108 * S.4 * y - 2/81 * y^4 - 1/162 * y = 0.$$

This suggests that

$$f6 := 9S.6 - 6xy^2 = \frac{4x^4y^3 + x^4 + xz}{yz}$$

$$f5 := 162S.5 - xy = \frac{x^4y}{z}$$

$$f4 := 2187S.4 + 2y^3 + 1 = \frac{x^3y^3 + x^3}{z}$$

$$f3 := 972S.3 - 10y^4 - 5y = \frac{4x^3y^4 + 5x^3y}{z}$$

would be better fractions to use, with $S.2$ and $S.1$ unnecessary. But then one could use

$$g3 := f3 - 4yf4 = \frac{x^3y}{z}$$

probably throw away $f5 = g3x$, use

$$g6 := f6 - 4g3y = \frac{x^4 + xz}{yz}$$

$$g4 := f4 - g3 * y^2 = \frac{x^3}{z},$$

and then throw away $g3$. So $g4$ corresponds to $f_{9,9}$ from above, and $g6$ corresponds to $f_{8,6}$.

```
F:=GF(17);
P<x,y,z>:=PolynomialRing(F,3,
          "weight",
          [1,0,0,
           5,6,6,
           3,6,0]);
f:=x^6+x^3*z-y^3*z^2;
I:=ideal<P|f>;
J:=Normalisation(I:FFMin:=true);
"J=",J;
G:=GroebnerBasis(J[1][1]);G;#G;

    $.1 + 13*$.3 + 11*$.4*$.8 + 9*$.8^4 + 15*$.8,
    $.2 + 3*$.3*$.8 + 7*$.4*$.8^2 + 15*$.8^5 + 13*$.8^2,
    $.3^2 + 15*$.3*$.8 + 3*$.4*$.8^5 + 8*$.4*$.8^2 + 2*$.8^11 + 10*$.8^8 +
        12*$.8^5,
    $.3*$.4 + 14*$.3 + $.4*$.8^4 + 8*$.4*$.8 + 4*$.8^10 + $.8^7 + $.8^4,
    $.3*$.5 + 2*$.5*$.8 + 2*$.6*$.8^3 + 3*$.7*$.8^8,
    $.3*$.6 + 3*$.5*$.8^2 + 13*$.6*$.8^4 + 4*$.6*$.8 + 12*$.7*$.8^9 +
        9*$.7*$.8^6 + 9*$.7*$.8^3,
    $.3*$.7 + 5*$.5 + 14*$.6*$.8^2 + 10*$.7*$.8^4,
    $.3*$.8^3 + $.3 + 8*$.4*$.8^4 + 10*$.4*$.8 + 5*$.8^7 + 11*$.8^4,
    $.3*$.9 + 10*$.7^3*$.8^4 + 4*$.7^3*$.8 + 8*$.8^4*$.9 + 4*$.8*$.9,
    $.4^2 + 9*$.4*$.8^3 + 3*$.4 + 8*$.8^9,
    $.4*$.5 + 10*$.5 + 10*$.6*$.8^2 + 6*$.7*$.8^7,
    $.4*$.6 + 13*$.5*$.8 + 16*$.6*$.8^3 + 3*$.6 + 7*$.7*$.8^8 + 12*$.7*$.8^5 +
        11*$.7*$.8^2,
    $.4*$.7 + 4*$.5*$.8^2 + 10*$.6*$.8 + 14*$.7*$.8^3,
    $.4*$.9 + 3*$.7^3*$.8^3 + 3*$.7^3 + 6*$.8^3*$.9 + 3*$.9,
    $.5^2 + 15*$.5*$.7*$.8 + 13*$.7^2*$.8^5,
    $.5*$.6 + 13*$.5*$.7*$.8^2 + 15*$.6*$.7*$.8 + $.7^2*$.8^6 + 4*$.7^2*$.8^3,
    $.5*$.7^2 + 15*$.8^4*$.9,
    $.5*$.8^3 + 13*$.5 + 4*$.6*$.8^2 + $.7*$.8^4,
    $.5*$.9 + 15*$.7^4*$.8 + 15*$.7*$.8*$.9,
    $.6^3 + 12*$.7^3*$.8^9 + $.7^3*$.8^6 + 9*$.7^3*$.8^3 + 9*$.7^3 +
        15*$.8^12*$.9 + 9*$.8^9*$.9 + 9*$.8^6*$.9 + $.8^3*$.9 + 9*$.9,
    $.6^2*$.7 + 4*$.7^3*$.8^7 + 3*$.7^3*$.8^4 + 13*$.7^3*$.8 + 8*$.8^7*$.9 +
        3*$.8^4*$.9 + 13*$.8*$.9,
    $.6^2*$.8 + $.6*$.7*$.8^3 + 15*$.6*$.7 + 4*$.7^2*$.8^8 + 16*$.7^2*$.8^5 +
        4*$.7^2*$.8^2,
    $.6*$.7^2 + 13*$.7^3*$.8^2 + 9*$.8^5*$.9 + 15*$.8^2*$.9,
    $.6*$.8*$.9 + 9*$.7^4*$.8^3 + 15*$.7^4 + 5*$.7*$.8^3*$.9 + 15*$.7*$.9,
    $.7^6 + $.7^3*$.9 + 16*$.8^3*$.9^2
]
```

Since there are two independent variables, this is beyond the scope of MAGMA's

`IntegralClosure` function.

6 My qth-power algorithm in M2

My implementation in my `QthPower` package produces a weighted strict affine P-algebra presentation, with $P := \mathbf{F}[y, z]$:

```
loadPackage "QthPower";
Macaulay2, version 1.4
i1 : loadPackage "QthPower";

i2 : wtR=matrix{{5,6,6},{3,6,0}};

i3 : Rq=ZZ/17[x,y,z,Weights=>entries weightGrevlex(wtR)];

i4 : Iq={x^6+x^3*z-y^3*z^2};

i5 : icq=qthIntegralClosure(wtR,Rq,Iq);
     toString(icq)

o6 = ({y^2*z,
       x*y^2*z,
       x^5+x^2*z,
       x^4*y+x*y*z,
       x^3*y^2,
       x^2*y^2*z},

      {p_0^2-p_2*p_5*p_6+p_4*p_6,
       p_0*p_1+p_0-p_3*p_5^2,
       p_0*p_2-p_5^2*p_6,
       p_0*p_3-p_4*p_5*p_6,
       p_0*p_4-p_1*p_6,
       p_1^2+p_1-p_5^3,
       p_1*p_2-p_4*p_5^2,
       p_1*p_3-p_0*p_5,
       p_1*p_4-p_2*p_5+p_4,
       p_2^2-p_0*p_5-p_3,
       p_2*p_3-p_1*p_6-p_6,
       p_2*p_4-p_3*p_5,
       p_3^2-p_2*p_6,
       p_3*p_4-p_5*p_6,
       p_4^2-p_0},

        (ZZ/17)[p_0, p_1, p_2, p_3, p_4, p_5, p_6],
       matrix {{ 10,   9,   8,   7,   5,   6,   6},
               { 6,   9,   6,   3,   3,   6,   0}})
```

```
i11 : R0=QQ[x,y,z,Weights=>entries weightGrevlex(wtR)];

i12 : I0={x^6+x^3*z-y^3*z^2};

i13 : ic0=rationalIntegralClosure(wtR,R0,I0);
      toString(ic0)


o14 = ({y^2*z,
        x*y^2*z,
        x^5+x^2*z,
        x^4*y+x*y*z,
        x^3*y^2,
        x^2*y^2*z},

       {p_0^2-p_2*p_5*p_6+p_4*p_6,
        p_0*p_1+p_0-p_3*p_5^2,
        p_0*p_2-p_5^2*p_6,
        p_0*p_3-p_4*p_5*p_6,
        p_0*p_4-p_1*p_6,
        p_1^2+p_1-p_5^3,
        p_1*p_2-p_4*p_5^2,
        p_1*p_3-p_0*p_5,
        p_1*p_4-p_2*p_5+p_4,
        p_2^2-p_0*p_5-p_3,
        p_2*p_3-p_1*p_6-p_6,
        p_2*p_4-p_3*p_5,
        p_3^2-p_2*p_6,
        p_3*p_4-p_5*p_6,
        p_4^2-p_0},

        QQ[p_0, p_1, p_2, p_3, p_4, p_5, p_6],

        matrix{{10, 9, 8, 7, 5, 6, 6},
               {6, 9, 6, 3, 3, 6, 0}})
```

7 My qth-power algorithm in MAGMA

My qth-power algorithm in MAGMA sees $q = 3$ as a bad prime, but then reconstructs the char 0 answer from those over $q = 5, 7, 11$, giving a presentation in terms of a $\mathbf{Q}[f_{6,6}, f_{6,0}]$-module basis ($f_{0,0} := 1, f_{5,3}, f_{7,3}, f_{8,6}, f_{9,9}, f_{10,6}$).

```
Loading "intclos_func061809.mag"
$.1^6 - $.2^3*$.3^2 + $.1^3*$.3
------------------------------------
PHI:
1 $.1^2*$.2^2*$.3
2 $.1^3*$.2^2
3 $.1^4*$.2 + $.1*$.2*$.3
4 $.1^5 + $.1^2*$.3
5 $.1*$.2^2*$.3
6 $.2^3*$.3
7 $.2^2*$.3^2
8 $.2^2*$.3
------------------------------------
PSIQ:
1 $.5
2 $.6
3 $.7
------------------------------------
IQ:
1 $.5^2 - $.1
2 $.4^2 - $.3*$.7
3 $.4*$.5 - $.6*$.7
4 $.3^2 - $.1*$.6 - $.4
5 $.3*$.4 - $.2*$.7 - $.7
6 $.3*$.5 - $.4*$.6
7 $.2^2 - $.6^3 + $.2
8 $.2*$.3 - $.5*$.6^2
9 $.2*$.4 - $.1*$.6
10 $.2*$.5 - $.3*$.6 + $.5
11 $.1^2 - $.3*$.6*$.7 + $.5*$.7
12 $.1*$.2 - $.4*$.6^2 + $.1
13 $.1*$.3 - $.6^2*$.7
14 $.1*$.4 - $.5*$.6*$.7
15 $.1*$.5 - $.2*$.7
------------------------------------
totaltime= 0.100 seconds
0.100
```

```
Loading file "qchar051109.mag"
Loading "closure_func040909.mag"
1111111111111111111111111111111111111111111111111111111111
q= 3
Delta= $.2^6*$.3^2 + $.2^3*$.3^2
WT_MATRIX_T= [
    [ 7, 5, 4, 3, 2, 6, 6 ],
    [ 3, 3, 0, 3, 0, 6, 0 ]
]
time for q= 3 is 0.060 seconds
modulus= 3
1111111111111111111111111111111111111111111111111111111111
q= 5
Delta= $.2^2*$.3^2
WT_MATRIX_T= [
    [ 10, 9, 8, 7, 5, 6, 6 ],
    [  6, 9, 6, 3, 3, 6, 0 ]
]
time for q= 5 is 0.010 seconds
1111111111111111111111111111111111111111111111111111111111
q= 7
Delta= $.2^2*$.3^2
WT_MATRIX_T= [
    [ 10, 9, 8, 7, 5, 6, 6 ],
    [  6, 9, 6, 3, 3, 6, 0 ]
]
time for q= 7 is 0.010 seconds
modulus= 35
[
    f_9_9^5*f_6_6 + f_9_9^5 + f_9_9^2*f_6_6*f_6_0 + f_9_9^2*f_6_0,
    f_9_9*f_6_6^3*f_6_0 + f_9_9*f_6_6^2*f_6_0,
    f_9_9^5 + f_9_9^2*f_6_6^2*f_6_0 + f_9_9^2*f_6_0,
    f_9_9^3*f_6_6^2 - f_6_6^2*f_6_0,
    f_9_9^4*f_6_6 - f_9_9*f_6_6^2*f_6_0 + f_9_9*f_6_6*f_6_0,
    f_6_6^4*f_6_0 + f_6_6^3*f_6_0,
    f_6_6^3*f_6_0^2 + f_6_6^2*f_6_0^2,
    f_6_6^3*f_6_0 + f_6_6^2*f_6_0
]
1111111111111111111111111111111111111111111111111111111111
q= 11
Delta= $.2^2*$.3^2
WT_MATRIX_T= [
    [ 10, 9, 8, 7, 5, 6, 6 ],
    [  6, 9, 6, 3, 3, 6, 0 ]
]
time for q= 11 is 0.020 seconds
```

```
modulus= 385
[
    f_5_3^2*f_6_6^2*f_6_0,
    f_5_3^3*f_6_6^2,
    f_5_3^4*f_6_6 + f_5_3*f_6_6*f_6_0,
    f_5_3^5 + f_5_3^2*f_6_0,
    f_5_3*f_6_6^2*f_6_0,
    f_6_6^3*f_6_0,
    f_6_6^2*f_6_0^2,
    f_6_6^2*f_6_0
]
1 f_5_3^2*f_6_6^2*f_6_0
2 f_5_3^3*f_6_6^2
3 f_5_3^4*f_6_6 + f_5_3*f_6_6*f_6_0
4 f_5_3^5 + f_5_3^2*f_6_0
5 f_5_3*f_6_6^2*f_6_0
6 f_6_6^3*f_6_0
7 f_6_6^2*f_6_0^2
8 f_6_6^2*f_6_0

newrelations= [
    f_5_3^2 - f_10_6,
    f_7_3^2 - f_8_6*f_6_0,
    f_7_3*f_5_3 - f_6_6*f_6_0,
    f_8_6^2 - f_10_6*f_6_6 - f_7_3,
    f_8_6*f_7_3 - f_9_9*f_6_0 - f_6_0,
    f_8_6*f_5_3 - f_7_3*f_6_6,
    f_9_9^2 - f_6_6^3 + f_9_9,
    f_9_9*f_8_6 - f_5_3*f_6_6^2,
    f_9_9*f_7_3 - f_10_6*f_6_6,
    f_9_9*f_5_3 - f_8_6*f_6_6 + f_5_3,
    f_10_6^2 - f_8_6*f_6_6*f_6_0 + f_5_3*f_6_0,
    f_10_6*f_9_9 - f_7_3*f_6_6^2 + f_10_6,
    f_10_6*f_8_6 - f_6_6^2*f_6_0,
    f_10_6*f_7_3 - f_5_3*f_6_6*f_6_0,
    f_10_6*f_5_3 - f_9_9*f_6_0
]
```

8 Timings

Here are rough timing comparisons, all done on the same departmental computer. Times vary somewhat, even on the same machine with the same problem, and certainly are machine dependent. But this clearly points out how dependent certain implementations are on the characteristic, and whether they are competitive with each other time-wise. (— means not applicable, ** means time and/or space resources were exhausted.)

char	*normal*	*normalP*	*icFrac −tions*	*icFracP*	*Normal− isation*	*Integral Closure*	*M2 Qth*	*Magma Qth*
0	0.1	− − −	0.1	− − −	586.	− − −	0.6	0.1
2	0.1	0.0	0.2	0.0	0.1	− − −	0.1	0.0
3	0.2	0.0	0.2	0.1	0.1	− − −	0.2	0.1
5	0.1	0.0	0.2	0.0	**	− − −	0.1	0.0
7	0.1	0.0	0.2	0.1	5.	− − −	0.1	0.0
11	0.1	0.1	0.2	0.6	161.	− − −	0.1	0.0
13	0.1	0.2	0.2	1.0	12.	− − −	0.1	0.0
17	0.1	0.4	0.2	4.	18.	− − −	0.2	0.0
19	0.1	1.0	0.2	5.	451.	− − −	0.2	0.0
23	0.1	2.6	0.2	11.	18.	− − −	0.2	0.0
29	0.1	10.	0.2	39.	18.	− − −	0.2	0.0
31	0.1	11.	0.2	42.	16.	− − −	0.2	0.0
37	0.1	29.	0.3	119.	16.	− − −	0.3	0.0
41	0.1	126.	0.3	217.	28.	− − −	0.2	0.0
101	0.1		0.2	**	16.	− − −	0.5	0.0
4001	0.1				15.	− − −		13.

The problems with `icFracP` and `normalP` are due to poorly thought-out computational implemetentations of the crucial step $f^q (mod\ I)$ of the qth-power method. But what causes the sporadic problems in `Normalisation` is unknown to me; and why It seems never to give an answer for $q = 5$ is a further mystery.

9 Comments

It should be troubling that:

- while the suggested P-module basis has 5 non-trivial elements, in addition to the 2 free variables in P, `normal` uses $3+3$ variables, `normalP` with reduction uses $2+2$, `icFractions` uses $2+3$, `icFracP` probably $3+3$, and `Normalisation` probably $6-2+3$;
- while the suggested presentation is as a strict P-algebra, `normal` produces a strict A-algebra presentation, `normalP` with reduction, a generic quotient ring, `integralClosure` a generic affine A-algebra presentation, `icFracP` only fractions (not even interreduced or reduced *mod* A), and `Normalisation` a generic quotient ring;
- both `icFracP` and `normalP` are overly dependent timewise on the characteristic, even in comparison to `QthPower`, which is necessarily dependent on the size of the characteristic;
- there is no thought that the monomial ordering on the input should induce a monomial ordering on the output;
- normal forms modulo the ideal of induced realtions are not really useful.

And is it easy to check that the elements of my P-module basis are in these various integral closures? If so how? from the fractions, from the presentation, or by doing some further computations? Using `normal` the presentation usually allows me to solve for what the fractions really should be. Then I can weight them, choose a nice monomial ordering, change rings, and compute a new Gröbner basis. Using `integralClosure`, even if I do all this I may not be satisifed.

$w_{1,1}yz - x^4 - xz = 0$ should really tell me that $NF(x^4 + xz) = w_{1,1}yz$, not that $NF(w_{1,1}yz) = x^4 + xz$. And $w_{1,1}^2 x - x^3 y - yz = 0$ should tell me that $yz/x = w_{1,1}^2 - x^2 y$ should be part of my P-module basis. What would tell me to rewrite it as $(x^5 + x^2 z)/(y^2 z)$, I don't know. Probably I would have to add it in first, then recompute the presentation.

Even a properly weighted version such as

```
R=QQ[f99,f86,x,y,z,MonomialOrder=>{Weights=>{9,8,5,6,6},
                                    Weights=>{9,6,3,6,0},
                                    Weights=>{1,1,1,0,0},
                                    Weights=>{1,1,0,0,0},
                                    Weights=>{1,0,0,0,0}}];
I=ideal(x^6+x^3*z-y^3*z^2,x^4+x*z-f86*y*z,x^3+z-f99*z);
J=saturate(saturate(I,ideal(z)),ideal(y));
G=transpose gens gb J
 | f99x-f86y        |
 | x3-f99z+z        |
 | f99f86-xy2-f86   |
 | f86x2-y2z        |
 | f99^2-y3-f99     |
 | f86^2x-f99yz     |
 | f86^2y-x2y2-f86x |
 | f86^3-y3z-f99z   |
```

still needs to be interpreted as needing $f86^2 - x^2y = f86x/y = f73$ before a normal form computation such as $NF(x^5 + x^2z) = f73y^2z$ would show that $(x^5 + x^2z)/(y^2z)$ was in the integral closure.