Why weights and type I presentations are important

1 An almost type I example

Consider the example

$$A := \overline{\mathbf{F}}_2[y,x]/\langle y^4 + yx + x^6\rangle,$$

which should be a curve of genus 3.

In other characteristics it is not in one-point form since $y^4 + x^6 = (y^2 + ix^3)(y^2 - ix^3)$. But in characteristic 2 $y^4 + x^6 = (y^2 + x^3)^2$. If we believe in weights, then $wt(x) := 4$ and $wt(y) := 6$, so $wt(y^2) = wt(x^3) = 12$ means $wt(y^2 + ax^3) < 12$ for some $a \in \overline{\mathbf{F}}_2$. In this case it would necessarily be that $wt(y^2+x^3) = 5$ since $(y^2+x^3)^2 = yx$ with $wt(yx) = wt(y)+wt(x) = 6+4 = 10$.

Defining $z := y^2 + x^3$ leads to the type I curve

$$B := \overline{\mathbf{F}}_2[z,x]/\langle z^4 + zx^2 + x^5\rangle.$$

There should be functions of all weights (pole orders) other than 1, 2, and 3. Let's see what the various implementations produce; in particular whether we can guess at the functions of weights 4, 5, 6, and 7 and the genus, 3.

2 Singular

```
                     SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-3
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   March 2011
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "normal.lib";
> ring r=2,(y,x),wp(6,4);
> ideal i=y4+yx+x6;
> list nor=normal(i);nor;
//   characteristic : 2
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    T(1)
//        block   2 : ordering wp
//                  : names    y x
//                  : weights  6 4
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=y3
      _[2]=x
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=y^4+x^6+y*x
s[2]=T(1)*x+y^3
s[3]=T(1)*y+x^5+y
s[4]=T(1)^2+T(1)+y^2*x^4
--------------------------------------
> ring r=2,(y,x),wp(6,4);
> ideal i=y4+yx+x6;
> list norp=normalP(i,"withRing");norp;
//   characteristic : 2
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    T(1)
//        block   2 : ordering wp
//                  : names    y x
//                  : weights  6 4
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=y3
```

```
      _[2]=x
[3]:
   [1]:
      1
   [2]:
      1
> def Rp=norp[1][1];
> setring Rp;
> option(redSB);
> ideal sp=std(norid);sp;
sp[1]=y^4+x^6+y*x
sp[2]=T(1)*x+y^3
sp[3]=T(1)*y+x^5+y
sp[4]=T(1)^2+T(1)+y^2*x^4
```

```
                    SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-3
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 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   March 2011
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "normal.lib";
> ring r=2,(z,x),wp(5,4);
> ideal i=z4+zx2+x5;
> list nor=normal(i);nor;
//   characteristic : 2
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    T(1) T(2)
//        block   2 : ordering wp
//                  : names    z x
//                  : weights  5 4
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=z2x
      _[2]=z3
      _[3]=x2
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=z^4+x^5+z*x^2
s[2]=T(2)*z+x^3+z
s[3]=T(2)*x^2+z^3
s[4]=T(1)*x+z^2
s[5]=T(1)*z+T(2)*x
s[6]=T(2)^2+T(2)+z^2*x
s[7]=T(1)*T(2)+T(1)+z*x^2
s[8]=T(1)^2+x^3+z
-----------------------------------------
> ring r=2,(z,x),wp(5,4);
> ideal i=z4+zx2+x5;
> list norp=normalP(i,"withRing");norp;
//   characteristic : 2
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    T(1) T(2)
//        block   2 : ordering wp
//                  : names    x
//                  : weights  4
//        block   3 : ordering C
```

```
[2]:
   [1]:
      _[1]=z2x
      _[2]=z3
      _[3]=x2
[3]:
   [1]:
      3
   [2]:
      3
> def Rp=norp[1][1];
> setring Rp;
> option(redSB);
> ideal sp=std(norid);sp;
sp[1]=T(2)^2+T(1)*x^2+T(2)
sp[2]=T(1)^2*x^2+T(1)*T(2)+T(1)+x^5
sp[3]=T(1)^2*T(2)+T(1)^2+T(2)*x^3
sp[4]=T(1)^3+T(1)*x^3+T(2)*x
```

3 Macaulay2

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "QthPower";

i2 : wtr=matrix{{6,4}};

i3 : R=ZZ/2[y,x,Weights=>entries weightGrevlex(wtr)];

i4 : I={y^4+y*x+x^6};

i5 : time ic=qthIntegralClosure(wtr,R,I);
     -- used 0.0216014 seconds

i6 : toString ic

o6 = ({x,
       y*x,
       y^2*x,
       y^3},
{    p_0^2+p_0+p_1*p_3^4,
     p_0*p_1+p_1+p_2*p_3^5,
     p_0*p_2+p_2+p_3^5,
     p_1^2+p_2*p_3+p_3^6,
     p_1*p_2+p_0*p_3,
     p_2^2+p_1},
 (ZZ/2)[p_0, p_1, p_2, p_3],
matrix {{14,  12,   6,   4}})

i7 : A=R/ideal(I);

i8 : time icf=icFractions A;
     -- used 0.0899194 seconds

i9 : toString icf

o9 = {(y^3)/(x),
       y,
       x}

i10 : time icp=icFracP A;
     -- used 0.0268694 seconds
```

```
i11 : toString icp

o11 = {1,
      (y^3)/(x)}

i12 : G=transpose gens gb presentation integralClosure A

o12 = {-6} | y4+x6+yx                 |
      {-3} | w_(0,0)x+y3              |
      {-5} | w_(0,0)y+x5+y            |
      {-6} | w_(0,0)^2+w_(0,0)+y2x4 |
```

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "QthPower";

i2 : wtr=matrix{{5,4}};

i3 : R=ZZ/2[z,x,Weights=>entries weightGrevlex(wtr)];

i4 : I={z^4+z*x^2+x^5};

i5 : time ic=qthIntegralClosure(wtr,R,I);
     -- used 0.0269921 seconds

i6 : toString ic

o6 = ({x^2,
       z*x^2,
       z^2*x,
       z^3},
{    p_0^2+p_0+p_1*p_3^2,
     p_0*p_1+p_1+p_2*p_3^2,
     p_0*p_2+p_2+p_3^3,
     p_1^2+p_2+p_3^3,
     p_1*p_2+p_0*p_3,
     p_2^2+p_1*p_3},
(ZZ/2)[p_0, p_1, p_2, p_3],
matrix {{7,   6,   5,   4}})

i7 : A=R/ideal(I);

i8 : time icf=icFractions A;
     -- used 0.13709 seconds

i9 : toString icf

o9 = {(z^2)/(x),
      (x^3+z)/(z),
       z,
       x}

i10 : time icp=icFracP A;
     -- used 0.0305596 seconds

i11 : toString icp
```

```
o11 = {1,
      (z^2)/(x),
      (x^3+z)/(z)}

i12 : G=transpose gens gb presentation integralClosure A

o12 = {-5} | z4+x5+zx2                     |
      {-3} | w_(1,1)z+x3+z                 |
      {-3} | w_(1,1)x2+z3                  |
      {-2} | w_(1,0)x+z2                   |
      {-2} | w_(1,0)z+w_(1,1)x             |
      {-3} | w_(1,1)^2+w_(1,1)+z2x         |
      {-3} | w_(1,0)w_(1,1)+w_(1,0)+zx2 |
      {-3} | w_(1,0)^2+x3+z                |
```

4 Magma

```
F:=GF(2);
P<y,x>:=PolynomialRing(F,2,"grevlexw",[6,4]);
I:=ideal<P|y^4+y*x+x^6>;
N:=Normalisation(I);
G:=GroebnerBasis(N[1][1]);G;

    $.1^5 + $.2*$.4 + $.2,
    $.1^4*$.3 + $.4^2 + $.4,
    $.1^3*$.2*$.3^2 + $.4^3 + $.4^2,
    $.1^3*$.3^3 + $.2*$.4^3 + $.2*$.4^2,
    $.1^2*$.3^4 + $.4^4 + $.4^3,
    $.1*$.2*$.3^5 + $.4^5 + $.4^4,
    $.1*$.3^6 + $.2*$.4^5 + $.2*$.4^4,
    $.1*$.4 + $.2*$.3,
    $.2^2 + $.3,
    $.3^7 + $.4^6 + $.4^5
```

```
F:=GF(2);
P<z,x>:=PolynomialRing(F,2,"grevlexw",[5,4]);
I:=ideal<P|z^4+z*x^2+x^5>;
N:=Normalisation(I);
G:=GroebnerBasis(N[1][1]);G;

    $.1^3 + $.2 + $.3^2,
    $.1^2*$.2 + $.3*$.4 + $.3,
    $.1*$.2^2 + $.4^2 + $.4,
    $.1*$.3 + $.2^2,
    $.1*$.4 + $.2*$.3,
    $.2^4 + $.3*$.4^2 + $.3*$.4,
    $.2^3*$.3 + $.4^3 + $.4^2,
    $.2^2*$.3^3 + $.4^4 + $.4^3,
    $.2*$.3^5 + $.4^5 + $.4^4,
    $.2*$.4 + $.3^2,
    $.3^7 + $.4^6 + $.4^5
```

5 qth-power in Magma

```
WT_MATRIX_T= [
    [ 14, 12, 6, 4 ]
]
I=
1 y_6^2 + y_12
2 y_12^2 + y_4^6 + y_6*y_4
3 y_12*y_6 + y_14*y_4
4 y_14^2 + y_12*y_4^4 + y_14
5 y_14*y_12 + y_6*y_4^5 + y_12
6 y_14*y_6 + y_4^5 + y_6
delta= x_4
phi=
1 x_6^3
2 x_6^2*x_4
3 x_6*x_4
4 x_4^2
psi=
1 y_6
2 y_4
```

```
WT_MATRIX_T= [
    [ 7, 6, 5, 4 ]
]
I=
1 y_5^2 + y_6*y_4
2 y_6^2 + y_4^3 + y_5
3 y_6*y_5 + y_7*y_4
4 y_7^2 + y_6*y_4^2 + y_7
5 y_7*y_6 + y_5*y_4^2 + y_6
6 y_7*y_5 + y_4^3 + y_5
delta= x_4^2
phi=
1 x_5^3
2 x_5^2*x_4
3 x_5*x_4^2
4 x_4^3
psi=
1 y_5
2 y_4
```

6 Comments

It should be clear from all of these that all implementations, including both of mine don't suggest there are functions of weights 5 and 7 in the non-type I example, while all do, at least implicitly, in the type I example. So, while none of these are technically wrong, the results in the non-type I case are certainly misleading in that they allow two functions of the same weight 12 to be used as variables. After all, we know that $wt(y^2) = 12 = wt(x^3)$, but that $wt(y^2+x^3) = 5$. And we know that $wt(y^3/x) = 14 = wt(yx^2)$, but that $wt(y^3/x + yx^2) = 7$. It's just that this information cannot be gotten directly from the presentation of $C(A, Q(A))$ though it is either implicit or explicit from $C(B, Q(B))$.