Annotated Examples 4

Generic example similar to example 3 but with rational numbers to reconstruct

This is almost like example 3, except that there are non-trivial rational numbers occurring as coefficients. So my algorithm must reconstruct these. Clearly one can't even read these in characteristic 2 or 17 because some denomiators vanish there. But there are a handful of characteristics, including 3 and 5, to avoid, mostly because the integral closure is slightly larger than what ones gets by specializing the result from characteristic 0.

$$\mathbf{F}[y;x]/\langle (y^2 - \frac{3}{4}y - \frac{15}{17}x)^3 - 9yx^4(y^2 - \frac{3}{4}y - \frac{15}{17}x) - 27x^{11}\rangle$$

My qth-power algorithm reconstructs the answer over the rationals from answers in characteristics 7,11,13,19,23,29, and 31, with this whole "wasteful" process still taking only about 3 seconds:

```
-------------------------------------------------
PHI:
1 $.1^5 - 9/4*$.1^4 - 30/17*$.1^3*$.2 + 27/16*$.1^3 + 45/17*$.1^2*$.2 +
    225/289*$.1*$.2^2 - 27/64*$.1^2 - 135/136*$.1*$.2 - 675/1156*$.2^2
2 $.1^3*$.2^3 - 15/17*$.1*$.2^4 - 9/16*$.1*$.2^3 - 45/68*$.2^4
3 $.1^4*$.2 - 3/2*$.1^3*$.2 - 30/17*$.1^2*$.2^2 + 9/16*$.1^2*$.2 +
    45/34*$.1*$.2^2 + 225/289*$.2^3
4 $.1*$.2^5
5 $.1^2*$.2^3 - 3/4*$.1*$.2^3 - 15/17*$.2^4
6 $.2^6
7 $.2^5
------------------------------------
PSIQ:
1 $.4
2 $.6
------------------------------------
IQ:
1 $.5^2 - $.3
2 $.4^2 - $.5*$.6^2 - 3/4*$.4 - 15/17*$.6
3 $.4*$.5 - $.2 + 3/4*$.5
4 $.3^2 - 27*$.5*$.6^5 - 9*$.1*$.6 - 27/4*$.3
5 $.3*$.4 - $.1*$.6 - 3/4*$.3
6 $.3*$.5 - 27*$.6^5 - 9*$.2 + 27/4*$.5
7 $.2^2 - 27*$.6^7 - 9*$.2*$.6^2 - 9/4*$.1*$.6 - 15/17*$.3*$.6 + 27/4*$.5*$.6^2
    - 9/4*$.3
8 $.2*$.3 - 27*$.4*$.6^5 - 9*$.3*$.6^2 - 81/4*$.6^5 - 27/2*$.2 - 135/17*$.5*$.6
    + 81/8*$.5
9 $.2*$.4 - $.3*$.6^2 - 3/2*$.2 - 15/17*$.5*$.6 + 9/8*$.5
```

```
10 $.2*$.5 - $.1*$.6 - 3/2*$.3
11 $.1^2 - 27*$.3*$.6^5 - 243*$.4*$.6^5 + 81/4*$.2*$.6^3 - 405/17*$.5*$.6^4 -
    81*$.3*$.6^2 + 729/4*$.6^5 - 243/8*$.5*$.6^3 - 135/17*$.1 - 1215/17*$.5*$.6
12 $.1*$.2 - 27*$.5*$.6^6 - 9*$.1*$.6^2 - 81/4*$.4*$.6^4 - 405/17*$.6^5 -
    27/2*$.3*$.6 + 243/16*$.6^4 - 135/17*$.2
13 $.1*$.3 - 27*$.2*$.6^4 - 243*$.6^6 + 81/2*$.5*$.6^4 - 81*$.2*$.6 - 135/17*$.3
    + 243/4*$.5*$.6
14 $.1*$.4 - 27*$.6^6 - 9*$.2*$.6 - 15/17*$.3 + 27/4*$.5*$.6
15 $.1*$.5 - 27*$.4*$.6^4 - 9*$.3*$.6 + 81/4*$.6^4 - 135/17*$.5
-----------------------------------
totaltime= 2.910 seconds
```

1 Magma

`IntegralClosure` gives $\mathbf{F}[f_6]$-module bases:

```
1 1
2 Y
3 1/X^2*Y^2 - 3/4/X^2*Y - 15/17/X
4 1/X^2*Y^3 + (-15/17*X - 9/16)/X^2*Y - 45/68/X
5 1/X^4*Y^4 - 3/2/X^4*Y^3 + (-30/17*X + 9/16)/X^4*Y^2 + 45/34/X^3*Y +
    225/289/X^2
6 1/X^5*Y^5 - 9/4/X^5*Y^4 + (-30/17*X + 27/16)/X^5*Y^3 + (45/17*X -
    27/64)/X^5*Y^2 + (225/289*X - 135/136)/X^4*Y - 675/1156/X^3
time for char=0 is 0.040
```

with implicit weights $(0, 11, 10, 21, 20, 25)$, as with the qth-power algorithm above.

2 Singular

The `normal` function gives:

```
                  SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-3
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   March 2011
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> system("--ticks-per-sec",100);
> LIB "normal.lib";
> ring r=0,(y,x),wp(11,6);
> ideal i=(y^2-3/4*y-15/17*x)^3-9*y*x^4*(y^2-3/4*y-15/17*x)-27*x^11;
> int time=timer;
> list nor=normal(i);nor;
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    T(1) T(2) T(3)
//        block   2 : ordering wp
//                  : names    y x
//                  : weights  11 6
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=68y2x3-51yx3-60x4
      _[2]=4624y4x-6936y3x-8160y2x2+2601y2x+6120yx2+3600x3
      _[3]=18496y5-41616y4-32640y3x+31212y3+48960y2x+14400yx2-7803y2-18360yx-10800x2
      _[4]=x5
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=314432*y^6-8489664*x^11-2829888*y^3*x^4-707472*y^5-832320*y^4*x+2122416*y^2*x^4
    +530604*y^4+2496960*y*x^5+1248480*y^3*x+734400*y^2*x^2-132651*y^3
    -468180*y^2*x-550800*y*x^2-216000*x^3
s[2]=68*T(3)*y^2*x-51*T(3)*y*x-60*T(3)*x^2-33958656*y*x^7-11319552*y^4+25468992*x^7
    +16979328*y^3+9987840*y^2*x-6367248*y^2-7490880*y*x
s[3]=3468*T(3)*y*x^3-340*T(3)*y^2+255*T(3)*y+300*T(3)*x-1731891456*x^9
    +169793280*y*x^6-577297152*y^3*x^2-127344960*x^6+432972864*y^2*x^2
    +509379840*y*x^3+49939200*y^2-37454400*y-44064000*x
s[4]=T(3)*x^5-18496*y^5+41616*y^4+32640*y^3*x-31212*y^3-48960*y^2*x-14400*y*x^2
    +7803*y^2+18360*y*x+10800*x^2
s[5]=272*T(3)*y^3-2448*T(3)*x^4-408*T(3)*y^2-240*T(3)*y*x+153*T(3)*y+180*T(3)*x
    -135834624*y^2*x^6+203751936*y*x^6-76406976*x^6-39951360*y^3+59927040*y^2
    +35251200*y*x-22472640*y-26438400*x
```

```
s[6]=60*T(2)*x-17*T(3)*y*x+8489664*x^7+2829888*y^3-2122416*y^2-2496960*y*x
s[7]=4*T(2)*y-3*T(2)-T(3)*x
s[8]=T(1)*x^2-68*y^2+51*y+60*x
s[9]=41616*T(1)*y*x+60*T(2)-17*T(3)*y+8489664*x^6
s[10]=4080*T(1)*y^2-3060*T(1)*y-3600*T(1)*x-17*T(3)*y*x^2+8489664*x^8
     +2829888*y^3*x-2122416*y^2*x-2496960*y*x^2
s[11]=5*T(3)^2-1797811200*T(1)*x-8489664*T(3)*y*x^2-734400*T(3)-46183772160*y^4*x
     +4239670284288*x^8-415653949440*y*x^5+1517136915456*y^3*x+40750387200*y^2*x^2
     +311740462080*x^5-1137852686592*y^2*x-1308087429120*y*x^2
     +19483778880*y*x+22922092800*x^2
s[12]=T(2)*T(3)-41616*T(3)*y-2309188608*y^3*x^2+3463782912*y^2*x^2
     +2037519360*y*x^3-1298918592*y*x^2-1528139520*x^3
s[13]=5*T(1)*T(3)-734400*T(1)-3468*T(3)*y*x+1731891456*x^7-169793280*y*x^4
     +577297152*y^3+127344960*x^4-432972864*y^2-509379840*y*x
s[14]=T(2)^2-31212*T(2)-10404*T(3)*x-577297152*y^2*x^3+432972864*y*x^3+509379840*x^4
s[15]=T(1)*T(2)-41616*T(1)*y-8489664*x^5
s[16]=T(1)^2-T(2)
> timer-time;
12
```

While this may be technically correct, it allows large integers to creep into the answer. Should that be an issue? After all, any linear transformation will give another technically correct answer. But maybe not everyone agrees with what would be lifted from the same problem with the same approach in (large) positive characteristic. And, obviously this answer is really over **Z**, not **Q**.

3 Macaulay2

icFractions and integralClosure used not to be able to do this problem.
Now:

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "QthPower";

i2 : wtr=matrix{{11,6}};

i3 : R=QQ[y,x,Weights=>entries weightGrevlex(wtr)];

i4 : I={(y^2-3/4*y-15/17*x)^3-9*y*x^4*(y^2-3/4*y-15/17*x)-27*x^11};

i5 : A=R/ideal(I);

i6 : time icf=icFractions A;
     -- used 0.400422 seconds

i7 : toString icf

o7 = {(18496*y^5-41616*y^4-32640*y^3*x+31212*y^3+48960*y^2*x+14400*y*x^2
          -7803*y^2-18360*y*x-10800*x^2)
     /(x^5),
      (68*y^2-51*y-60*x)
     /(x^2),
       y,
       x}

i8 : G=transpose gens gb presentation integralClosure A

o8 = {-11} | 314432y6-8489664x11-2829888y3x4-707472y5-832320y4x+2122416y2x4+530
     {-2}  | w_(3,0)x2-68y2+51y+60x
     {-9}  | 4624w_(3,0)y4-6936w_(3,0)y3-8160w_(3,0)y2x+2601w_(3,0)y2+6120w_(3,
     {-7}  | 68w_(3,0)^2y2-51w_(3,0)^2y-60w_(3,0)^2x-8489664x7-2829888y3+212241
     {-5}  | w_(3,0)^3-41616w_(3,0)y-8489664x5
     {-1}  | w_(4,0)x-4w_(3,0)^2y+3w_(3,0)^2
     {-6}  | 17w_(4,0)y-60w_(3,0)^2-41616w_(3,0)yx-8489664x6
     {-5}  | w_(4,0)w_(3,0)-2448w_(3,0)^2x-146880w_(3,0)-33958656yx4+25468992x4
     {-6}  | w_(4,0)^2-146880w_(4,0)-407503872w_(3,0)y2+305627904w_(3,0)y-92367
     ------------------------------------------------------------------------
     604y4+2496960yx5+1248480y3x+734400y2x2-132651y3-468180y2x-550800yx2
```

```
0)yx-8489664x9-2829888y3x2+2122416y2x2+2496960yx3+244800y2-183600y-
6y2+2496960yx



54432y4x-83130789888yx5+20782697472y3x+8150077440y2x2+62348092416x5
----------------------------------------------------------------------
-216000x3                                                     |
                                                              |
216000x                                                       |
                                                              |
                                                              |
                                                              |
                                                              |
                                                              |
-15587023104y2x-12225116160yx2+3896755776yx+4584418560x2 |
```

This seems to have the same problem as SINGULAR's for much the same reasons, though it uses only ring generators and a slightly different product monomial ordering, lex-over-input.

4 Magma

`Normalisation` gives:

```
-$.1^2*$.2 - 15/17*$.1 + $.3^2 - 3/4*$.3,
 -$.1^2*$.2^2 - 15/17*$.1*$.2 + $.2*$.3^2 - 3/4*$.2*$.3,
 -$.1*$.4 + $.2^2*$.3 - 3/4*$.2^2,
 -27*$.1^6 - 9*$.1*$.2*$.3 - 15/17*$.2^2 + $.3*$.4,
 -27*$.1^5 + $.2^3 - 9*$.2*$.3,
 -27*$.1^4*$.3 + 81/4*$.1^4 - 9*$.1*$.2^2 + $.2*$.4 - 135/17*$.2,
 -27*$.1^7 - 9*$.1^2*$.2*$.3 - 15/17*$.1*$.2^2 - 9/4*$.1*$.4 + $.2^2*$.3^2 +
     3/2*$.2^2*$.3 - 27/16*$.2^2,
 -27*$.1^5*$.3 - 81/4*$.1^5 - 9*$.1^2*$.2^2 - 135/17*$.1*$.2 + $.2^3*$.3 +
     3/4*$.2^3 - 27/2*$.2*$.3,
 -27*$.1^6*$.2 - 405/17*$.1^5 - 81/4*$.1^4*$.3 + 243/16*$.1^4 - 9*$.1^2*$.4 -
     27/2*$.1*$.2^2 + $.2*$.3*$.4 - 135/17*$.2*$.3 + 3/4*$.2*$.4 -
     405/68*$.2,
 -27*$.1^5*$.2 - 9*$.1*$.4 + $.2^4 - 27/4*$.2^2,
 -243*$.1^6 - 27*$.1^4*$.2*$.3 + 81/4*$.1^4*$.2 - 81*$.1*$.2*$.3 + $.2^2*$.4
     - 135/17*$.2^2,
 -27*$.1^5*$.2^2 - 243*$.1^5*$.3 + 729/4*$.1^5 - 405/17*$.1^4*$.2 +
     81/4*$.1^3*$.2*$.3 - 243/16*$.1^3*$.2 - 81*$.1^2*$.2^2 - 1215/17*$.1*$.2
     + $.4^2 - 135/17*$.4
```

but there are over 4000 lines in the corresponding lex Gröbner basis.

Even reversing the order of the variables before producing a lex answer would have given a readable answer:

```
d^2 - 135/17*d - b^5 + 9/4*b^3*a^3 + 27/4*b^3 - 405/17*b*a^4 - 243/16*b*a^3
    - 243/4*a^8,
d*c - b^3*a - 15/17*b^2,
d*b - 27*c*a^4 - 9*b^2*a - 135/17*b + 81/4*a^4,
d*a - 1/9*b^4 + 3/4*b^2 + 3*b*a^5,
c^2 - 3/4*c - b*a^2 - 15/17*a,
c*b - 1/9*b^3 + 3*a^5,
c*a^5 - 1/243*b^5 + 1/36*b^3 + 1/9*b^2*a^5 + 1/3*b^2*a^2 + 5/17*b*a -
    3/4*a^5,
b^6 - 27/4*b^4 - 54*b^3*a^5 - 81*b^3*a^2 - 1215/17*b^2*a + 729/4*b*a^5 +
    729*a^10
```

which tells me that no thought was given whatsoever as to what might constitute a reasonable monomial ordering in such problems. In particular, one should try to get inspiration from the monomial ordering of the original ring, given that the integral closure lives in its quotient ring.

5 Qth-Power

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "QthPower";

i2 : wtr=matrix{{11,6}};

i3 : R=QQ[y,x,Weights=>entries weightGrevlex(wtr)];

i4 : I={(y^2-3/4*y-15/17*x)^3-9*y*x^4*(y^2-3/4*y-15/17*x)-27*x^11};

i5 : time ic=rationalIntegralClosure(wtr,R,I);
     -- used 14.9765 seconds

i6 : toString ic

o6 = ({x^5,
       y^2*x^3-(3/4)*y*x^3-(15/17)*x^4,
       y*x^5,
       y^4*x-(3/2)*y^3*x-(30/17)*y^2*x^2+(9/16)*y^2*x+(45/34)*y*x^2+(225/289)*x^3,
       y^3*x^3-(15/17)*y*x^4-(9/16)*y*x^3-(45/68)*x^4,
       y^5-(9/4)*y^4-(30/17)*y^3*x+(27/16)*y^3+(45/17)*y^2*x+(225/289)*y*x^2
          -(27/64)*y^2-(135/136)*y*x-(675/1156)*x^2},
{    p_0^2-(135/17)*p_0+(81/4)*p_1*p_5^3-27*p_2*p_5^5-81*p_2*p_5^2-243*p_3*p_5^5
        -(405/17)*p_4*p_5^4-(243/8)*p_4*p_5^3-(1215/17)*p_4*p_5+(729/4)*p_5^5,
     p_0*p_1-9*p_0*p_5^2-(135/17)*p_1-(27/2)*p_2*p_5-(81/4)*p_3*p_5^4
        -27*p_4*p_5^6-(405/17)*p_5^5+(243/16)*p_5^4,
     p_0*p_2-27*p_1*p_5^4-81*p_1*p_5-(135/17)*p_2+(81/2)*p_4*p_5^4
        +(243/4)*p_4*p_5-243*p_5^6,
     p_0*p_3-9*p_1*p_5-(15/17)*p_2+(27/4)*p_4*p_5-27*p_5^6,
     p_0*p_4-9*p_2*p_5-27*p_3*p_5^4-(135/17)*p_4+(81/4)*p_5^4,
     p_1^2-(9/4)*p_0*p_5-9*p_1*p_5^2-(15/17)*p_2*p_5-(9/4)*p_2
        +(27/4)*p_4*p_5^2-27*p_5^7,
     p_1*p_2-(27/2)*p_1-9*p_2*p_5^2-27*p_3*p_5^5-(135/17)*p_4*p_5
        +(81/8)*p_4-(81/4)*p_5^5,
     p_1*p_3-(3/2)*p_1-p_2*p_5^2-(15/17)*p_4*p_5+(9/8)*p_4,
     p_1*p_4-p_0*p_5-(3/2)*p_2,
     p_2^2-9*p_0*p_5-(27/4)*p_2-27*p_4*p_5^5,
     p_2*p_3-p_0*p_5-(3/4)*p_2,
     p_2*p_4-9*p_1+(27/4)*p_4-27*p_5^5,
     p_3^2-(3/4)*p_3-p_4*p_5^2-(15/17)*p_5,
     p_3*p_4-p_1+(3/4)*p_4,
     p_4^2-p_2},
```

```
      QQ[p_0, p_1, p_2, p_3, p_4, p_5],
matrix {{25,  21,  20,  11,  10,   6}})
```

6 Timings

Here are rough timing comparisons, all done on the same departmental computer. Times vary somewhat, even on the same machine with the same problem, and certainly are machine dependent. But this clearly points out how dependent certain implementations are on the characteristic, and whether they are competitive with each other time-wise. (* means the integral closure is bigger in this char, — means not applicable (or avoided here because it was a factor one some coefficient), *** means time and/or space resources were exhausted.)

char	*normal*	*normalP*	*icFrac −tions*	*icFracP*	*Normal− isation*	*Integral Closure*	*M2 Qth*	*Magma Qth*
0	0.2	− − −	0.8	− − −	189.	0.04	15.	3.
2	− − −	− − −	− − −	− − −	− − −	− − −	− − −	− − −
3	− − −	− − −	− − −	− − −	− − −	− − −	− − −	− − −
5	− − −	− − −	− − −	− − −	− − −	− − −	− − −	− − −
7	0.1	* * *	0.4	* * *	0.9	0.08	0.5	0.1
11	0.1		0.4		0.9	0.03	0.9	0.2
13	0.1		0.4		0.9	0.04	1.	0.2
17	− − −	− − −	− − −	− − −	− − −	− − −	− − −	− − −
19	0.1		0.4		1.	0.03	2.	0.3
23	0.1		0.4		1.	0.04	3.	0.5
29	0.1		0.4		1.	0.06	4.	0.7
31	0.1		0.4		1.	0.09	5.	0.9
37	0.1		0.4		1.	0.02	11.	0.9
41	0.1		0.5		1.	0.03	13.	1.
101	0.1		0.7		1.	0.2	75.	6.
4001	0.1		0.6		1.	0.03	* * *	* * *

[The 189 seconds for `Normalisation` is almost all used to produce the ridiculous Gröbner basis.] I do not know whether it is my fault of M2's that my `QthPower` runs much slower than the Magma version in general–undoubtedly mine; but it also takes me an order of magnitude more time to program this stuff in M2 as well–again undoubtedly my fault.