Local monomial orderings for integral closures of ideals

1 Simple examples of integral closures of ideals

The simplest type of example that gets across what integral closures of ideals are about is:

$$I_m := \langle y^m,\ x^m \rangle \subset \mathbf{F}[y, x] =: R,$$

having integral closure

$$C(I_m, R) := \langle y^m,\ y^{m-1}x,\ \ldots,\ yx^{m-1},\ x^m \rangle$$

generated by all the monomials lying “between” y^m and x^m.

But such monomial ideals rarely give insight into any monomial ordering, so try binomial ideals such as:

$$I_{m,n} := \langle y^m - y^n,\ x^m - x^n \rangle \subset \mathbf{F}[y, x] =: R$$

with $m > n$. Probably one can guess that the integral closure is related to either $C(I_m, R)$ or to $C(I_n, R)$. To see which try out `normalI` in SINGULAR and `integralClosure` in MACAULAY2.

```
                     SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-5
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   Jul 2012
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "normal.lib";
> ring r=0,(y,x),dp;
> ideal i=y3-y2,x3-x2;
> list nor=normalI(i);nor;
[1]:
   _[1]=y3-y2
   _[2]=x3-x2
   _[3]=-y2x2+y2x+yx2-yx
> ideal s=std(nor[1]);s;
s[1]=x3-x2
s[2]=y3-y2
s[3]=y2x2-y2x-yx2+yx
```

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : R=QQ[y,x];
i2 : I=ideal(y^3-y^2,x^3-x^2);
i3 : time IC=integralClosure(I);
```

```
     -- used 0.243466 seconds
i4 : toString IC
o4 = ideal(x^3-x^2,
           y^3-y^2,
           y^2*x^2-y^2*x-y*x^2+y*x)
```

It should be clear from these that y^2x^2 is not "between" y^3 and x^3, but that yx is between y^2 and x^2.

2 Local monomial orderings

From this it should be clear that the triling entries are more important to an understanding of the integral closure above than the leading entries.

```
ring r=0,(y,x),ds;
> ideal i=y3-y2,x3-x2;
> list nor=normalI(i);nor;
[1]:
   _[1]=-y2+y3
   _[2]=-x2+x3
   _[3]=-yx+y2x+yx2-y2x2
> ideal s=std(nor[1]);s;
s[1]=y2
s[2]=yx
s[3]=x2
```

While we might have expected what `normalI` gave us, perhaps we (meaning at least the royal I) were not ready for what standard (Gröbner) bases look like relative to local monomial orderings. Here $x^2 - x^3 = x^2u_1$ and $y^2 - y^3 = y^2u_2$ with $u_1 := 1 - x$ and $u_2 := 1 - y$ polynomial units. So the extra generator found is really of the form yxu_1u_2. The lesson here is that interreduction is much trickier with a local monomial ordering in that $x^2 - x^3$ could be reduced to $x^2 - x^s$ for any $s > 2$. So saying that $x^2 \in I$ above means only that $x^2u \in I$ for some polynomial unit u.

MACAULAY2 is not really set up to compute integral closures for local monomial orderings, so instead the following gives a local answer based on the global output.

```
 5 : S=QQ[y,x,MonomialOrder=>{Weights=>{-1,-1}},Global=>false];

i6 :    phi=map(S,R,matrix{{y,x}});
i7 :    ic=(flatten entries gens IC)/phi;
i8 : toString gens gb ideal ic
o8 = matrix {{x^2-x^3,
              y*x-y^2*x-y*x^2+y^2*x^2,
              y^2-y^3}}
```

3 Qth-power approach

My Macaulay2 code based on the Qth-power algorithm for integral closures of rings, uses a local monomial ordering, but is restricted currently to positive characteristic, and probably runs well only for small $Q > 0$. That said,

```
i3 :      R=ZZ/13[x,y,MonomialOrder=>{
                       Weights=>{-1,-1},
                       Weights=>{0,-1}},
                       Global=>false];
i4 :    I={x^2-x^3,y^2-y^3};
i5 :    wt={{-1,-1},{0,-1}};

i6 :      time IC=idealClosure(R,{0},I,wt);
  2    3   2    3
{x  - x , y  - y }
{1}
{x, y}
  2       2
{x , x*y, y }
  2       2
{x , x*y, y }
{1}
{x, y}
  2       2
{x , x*y, y }
     -- used 0.2507 seconds
i7 :  toString IC
o7 = {x^2, x*y, y^2}
```

does produce the local answer I'm advocating theoretically.