Local monomial orderings for integral closures of ideals

1 A slightly harder but generic example

Try a slightly harder but generic example (in characteristic 2 so coefficients aren't a problem):

$$I := \langle y^5 + x^5 + y^3x + x^2,\ y^6 + y^5x + x^6 + y^5 + y^4x\rangle \subset R := \mathbf{F}_2[y, x].$$

with global Gröbner basis

$$B := \{y^5 + x^5 + y^3x + x^2,$$
$$yx^5 + y^3x^2 + x^5 + y^3x + yx^2 + x^3 + x^2,$$
$$x^{10} + y^4x^4 + y^4x^3 + x^7 + y^3x^3 + y^2x^4 + y^4x + y^2x^3 + yx^3 + x^4 + yx^2\}.$$

A global answer from `integralClosure` is

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
i1 :      R=ZZ/2[y,x]
i2 :      I=ideal(y^5+x^5+y^3*x+x^2,y^6+y^5*x+x^6+y^5+y^4*x);
i3 :      G=gens gb I
o3 = | y5+x5+y3x+x2
       yx5+y3x2+x5+y3x+yx2+x3+x2
       x10+y4x4+y4x3+x7+y3x3+y2x4+y4x+y2x3+yx3+x4+yx2 |

i4 :      time ic1=integralClosure(I,1);
     -- used 0.302718 seconds
i5        G1=gens gb ic1;
i6 :      toString G1
o6 = matrix {{
        y^5+x^5+y^3*x+x^2,
        y*x^5+y^3*x^2+x^5+y^3*x+y*x^2+x^3+x^2,
        y^4*x^4+y^3*x^4+y^4*x^2+y^3*x^3+x^6+y^3*x^2+x^5+x^4+x^2,
        x^9+y^4*x^3+y^3*x^4+y^4*x^2+y^4*x+y^2*x^3+y*x^4+x^5+x^2}}

i7 :      time ic2=integralClosure(I,2);
     -- used 0.249508 seconds
i8 :      G2=gens gb ic2;
i9 :      toString G2

o9 = matrix {{
     y^10+x^10+y^6*x^2+x^4,
     y^6*x^5+y*x^10+y^8*x^2+y^5*x^5+y^4*x^6+y^3*x^7+x^10+y^8*x+y^6*x^3+y^5*x^3+
```

```
          x^8+y^5*x^2+y^4*x^3+y*x^4+x^5+x^4,
     y^2*x^10+y^6*x^4+x^10+y^6*x^2+y^2*x^4+x^6+x^4,
     y^9*x^4+y^4*x^9+y^8*x^4+y^3*x^9+y^8*x^3+y^3*x^8+y*x^10+x^11+y^9*x+y^8*x^2+
          y^5*x^5+y^4*x^6+y^5*x^4+y*x^8+x^9+y^5*x^3+y^4*x^4+x^8+y^5*x^2+y^3*x^4+
          y^3*x^3+y*x^5+y*x^4,
     x^14+y^8*x^4+y^4*x^8+y^9*x^2+y^4*x^7+x^11+y^8*x^2+y^6*x^4+y^3*x^7+y^2*x^8+
          y*x^9+y^8*x+y^6*x^3+y^5*x^4+y^4*x^4+y^3*x^5+x^8+y^3*x^4+y^2*x^5+x^5,
     y^5*x^9+y^9*x^3+y^3*x^9+y^9*x+y^5*x^3+y^3*x^5+y^3*x^3}}

i10 :     time ic3=integralClosure(I,3);
     -- used 0.27486 seconds
i11 :     G3=gens gb ic3;
i12 :     toString G3

o12 = matrix {{
      y^15+y^10*x^5+y^5*x^10+x^15+y^13*x+y^3*x^11+y^11*x^2+y^6*x^7+
           y^10*x^2+y^9*x^3+x^12+y^6*x^4+y^5*x^4+x^9+y^3*x^5+x^6,
      y^11*x^5+y*x^15+y^13*x^2+y^10*x^5+y^3*x^12+x^15+y^13*x+y^7*x^7+y^3*x^11+
           y^11*x^2+y^10*x^3+y^9*x^4+y^6*x^7+y*x^12+x^13+y^10*x^2+y^9*x^3+
           x^12+y^7*x^4+y^6*x^5+y^6*x^4+y*x^9+y^3*x^6+x^9+y^3*x^5+y*x^6+x^7+x^6,
      y^7*x^10+y^2*x^15+y^5*x^11+y^11*x^4+y^6*x^9+y^5*x^10+x^15+y^9*x^5+
           y^3*x^11+y^2*x^12+y^11*x^2+y^6*x^7+y^9*x^3+y^6*x^6+x^12+y^7*x^4+
           y^5*x^6+y^2*x^9+x^11+y^6*x^4+y^5*x^5+y^3*x^7+y^5*x^4+x^9+y^3*x^5+
           y^2*x^6+x^8+x^6,
      y^3*x^15+y^5*x^12+y^2*x^15+y^7*x^9+y^5*x^11+y*x^15+y^9*x^6+y^6*x^9+
           y^2*x^13+x^15+y^9*x^5+y^7*x^7+y^3*x^11+y^2*x^12+y^9*x^4+y^7*x^6+
           y*x^12+x^13+y^9*x^3+y^6*x^6+y^3*x^9+y*x^11+x^12+y^7*x^4+y^6*x^5+
           y^5*x^6+y^3*x^8+y^2*x^9+x^11+y^6*x^4+y^5*x^5+y^3*x^7+y*x^9+y^2*x^7+
           y*x^8+y^3*x^5+y^2*x^6+x^8+y*x^6+x^7+x^6,
      y^14*x^4+y^4*x^14+y^13*x^4+y^3*x^14+y^14*x^2+y^13*x^3+y^4*x^12+y^3*x^13+
           x^16+y^13*x^2+y^10*x^5+y^9*x^6+y^3*x^12+x^15+y^9*x^5+y^6*x^8+x^14+
           y^9*x^4+y^6*x^7+y^10*x^2+y^6*x^6+y^4*x^8+x^12+y^3*x^8+y^6*x^4+
           y^4*x^6+y^3*x^7+x^10+y^3*x^6+x^9+x^8+x^6,
      y^5*x^14+x^19+y^13*x^4+y^9*x^8+y^8*x^9+y^5*x^12+y^4*x^13+y^2*x^15+
           y^14*x^2+y^4*x^12+y^12*x^3+y^11*x^4+y^10*x^5+y^9*x^6+y^6*x^9+
           y^3*x^12+y*x^14+y^10*x^4+y^2*x^12+y^10*x^3+y^9*x^4+y^7*x^6+y^6*x^7+
           y^5*x^8+y^3*x^10+y^10*x^2+y^6*x^6+y^8*x^3+y^5*x^6+y^3*x^8+y^2*x^9+
           x^11+y^4*x^6+y^3*x^6+y*x^8+x^9+y^2*x^6+x^8+x^7,
      y^10*x^9+x^19+y^14*x^3+y^13*x^4+y^6*x^11+y^4*x^13+y^3*x^14+y^14*x^2+
           y^4*x^12+y^14*x+y^12*x^3+y^11*x^4+y^9*x^6+y^4*x^11+y^2*x^13+y*x^14+
           x^15+y^10*x^4+y^10*x^3+y^8*x^5+y^7*x^6+y^6*x^7+x^13+y^10*x^2+x^12+
           y^4*x^7+y^3*x^8+y^6*x^4+y^4*x^6+y^4*x^5+y^2*x^7+y*x^8+x^9+x^6,
      x^20+y^2*x^17+y^6*x^12+y^4*x^14+y^14*x^3+y^10*x^7+y^6*x^11+y^14*x^2+
           y^13*x^3+y^10*x^6+y^6*x^10+y^2*x^14+y^14*x+y^12*x^3+y^10*x^5+
           y^8*x^7+y^4*x^11+y^12*x^2+y^11*x^3+y^9*x^5+y^7*x^7+y^11*x^2+y^10*x^3+
           y^2*x^11+y*x^12+x^13+y^4*x^8+y^7*x^4+y^2*x^9+y^6*x^4+y^2*x^8+y*x^9+
```

```
      y^4*x^5+x^8+y*x^6,
  y*x^19+x^19+y^9*x^9+y^2*x^16+y^10*x^7+y^8*x^9+y^5*x^12+y^4*x^13+
      y^2*x^15+x^17+y^13*x^3+y^12*x^4+y^10*x^6+y^9*x^7+y^6*x^10+y^5*x^11+
      y^3*x^13+y^12*x^3+y^11*x^4+y^10*x^5+y^9*x^6+y^3*x^12+y^2*x^13+
      y*x^14+y^13*x+y^11*x^3+y^8*x^6+y^7*x^7+y^6*x^8+y^2*x^12+x^14+
      y^9*x^4+y^7*x^6+y^10*x^2+y^9*x^3+y^5*x^7+y^3*x^9+y^2*x^10+x^12+
      y^8*x^3+y^5*x^6+y^3*x^8+y^2*x^9+y^5*x^5+y*x^9+x^10+y^3*x^6+y^2*x^7+
      y*x^8+y^2*x^6+y*x^7+x^7}}
```

Of course, the integral closure of the Rees algebra

```
i13 :     reesi=(flattenRing reesAlgebra(I))_0
i14 :     rbar=integralClosure(reesi);

i15 :     gbar=gens gb ideal rbar;

i16 :     toString gbar

o16 = matrix {{
      w_0*(y^6+y^5*x+x^6+y^5+y^4*x)+w_1*(y^5+x^5+y^3*x+x^2),
      w_(1,0)*x+w_0*(y^4+y^3)+w_1*y^3,
      w_(1,0)*y^2+w_0*(y^5+x^5+y^4)+w_1*(x^4+y^3+x),
      w_(1,0)^2+w_(1,0)*w_1*y
         +w_0^2*(y^5*x+y^2*x^4+y*x^5+x^6+y^5+y^4*x+y*x^4+x^5+y^4)
         +w_0*w_1*(y^5+y^2*x^3+x^5+y^4+y^3*x+y*x^3+x^4+y^3+y^2+y*x+x^2+y+x)
         +w_1^2*(y*x^3+*y)}}
```

seems to suggest that $C(I,R)$ should suffice to determine $C(I^k,R)$ for all $k \geq 0$. Let us see this by mapping to a ring with a local monomial ordering, as before.

```
i19 : S=ZZ/2[y,x,MonomialOrder=>{
                      Weights=>{-1,-1},
                      Weights=>{0,-1}},
                      Global=>false];

i20 :     phi=map(S,R,matrix{{y,x}});

i21 :     g1=(flatten entries G1)/phi;
i22 :     g2=(flatten entries G2)/phi;
i23 :     g3=(flatten entries G3)/phi;
i24 : h1=gens gb ideal g1;
i25 :  toString h1
o25 = matrix {{
      y^5+y^4*x+y^6+y^5*x+x^6,
      y^3*x+x^4+y^5+y^3*x^2+y^4*x^2+y^3*x^3+x^6+y^3*x^4+y^4*x^4,
      x^2+y^3*x+y^5+x^5}}
```

```
i26 : h2=gens gb ideal g2;
i27 :  toString h2
o27 = matrix {{
      y^10+y^8*x^2+y^12+y^10*x^2+x^12,
      y^8*x+y^5*x^4+y^10+y^6*x^4+y^3*x^7+y^2*x^8+y*x^9+y^10*x+
            y^8*x^3+y^4*x^7+y^12+y^9*x^3+y^8*x^4+y^6*x^6+y^4*x^8+
            x^12+y^13+y^12*x+y^3*x^10+y^2*x^11+y^14+y^13*x+y^10*x^4+
            y^4*x^10+y^3*x^11,
      y^5*x^2+y^4*x^3+y^6*x^2+y^5*x^3+x^8+y^8*x+y^7*x^2+y^10+
            y^8*x^2+y^5*x^5+y^4*x^6+y^3*x^7+y^11+y^10*x+y^6*x^5+x^11,
      y^3*x^3+y*x^5+y^5*x^2+y^3*x^4+y^5*x^3+y^4*x^4+x^8+y^7*x^2+
            y^5*x^4+y*x^8+x^9+y^9*x+y^8*x^2+y^5*x^5+y^4*x^6+y^11+
            y^8*x^3+y^3*x^8+x^11+y^8*x^4+y^3*x^9+y^9*x^4+y^4*x^9,
      x^4+y^6*x^2+y^10+x^10}}

i28 : h3=gens gb ideal g3;
i29 :  toString h3

o29 = matrix {{
      y^15+y^14*x+y^13*x^2+y^12*x^3+y^16+y^15*x+y^14*x^2+y^13*x^3+y^10
           *x^6+y^8*x^8+y^17+y^16*x+y^15*x^2+y^14*x^3+y^5*x^12+y^4*x^13
           +y^18+y^17*x+y^16*x^2+y^15*x^3+y^12*x^6+y^10*x^8+y^6*x^12
           +y^5*x^13+x^18,
      y^13*x+y^11*x^3+y^10*x^4+y^8*x^6+y^14*x+y^13*x^2+y^12*x^3+y^11*x^4
           +y^16+y^13*x^3+y^11*x^5+y^10*x^6+y^3*x^13+x^16+y^16*x+y^15*x^2
           +y^14*x^3+y^11*x^6+y^4*x^13+y^3*x^14+y^18+y^17*x+y^13*x^5
           +y^12*x^6+y^6*x^12+y^5*x^13+y^4*x^14+y^3*x^15+y^15*x^4+y^13*x^6
           +y^3*x^16+y^16*x^4+y^14*x^6+y^4*x^16,
      y^10*x^2+y^8*x^4+y^13*x+y^12*x^2+y^11*x^3+y^10*x^4+x^14+y^15+y^13*x^2
           +y^10*x^5+y^8*x^7+y^15*x+y^13*x^3+y^3*x^13+y^17+y^15*x^2+y^12*x^5
           +y^10*x^7+y^5*x^12+x^17,
      y^8*x^3+y^7*x^4+y^6*x^5+y^4*x^7+y^10*x^2+y^8*x^4+y^7*x^5+y^5*x^7
           +y^3*x^9+y*x^11+x^12+y^9*x^4+y^8*x^5+y^4*x^9+y^12*x^2+y^10*x^4
           +x^14+y^12*x^3+y^11*x^4+y^10*x^5+y^8*x^7+x^15+y^13*x^3+y^11*x^5
           +y^9*x^7+y^7*x^9+y^6*x^10+y^17+y^12*x^5+y^11*x^6+y^8*x^9+y^7*x^10
           +y^6*x^11+y^3*x^14+x^17+y^17*x+y^15*x^3+y^13*x^5+y^11*x^7
           +y^10*x^8+y^7*x^11+y^5*x^13+y^3*x^15+y*x^17+x^18+y^19+y^18*x
           +y^15*x^4+y^14*x^5+y^13*x^6+y^10*x^9+y^9*x^10+y^8*x^11+y^4*x^15
           +y^3*x^16,
      y^5*x^4+y^4*x^5+y^6*x^4+y^5*x^5+x^10+y^11*x^2+y^10*x^3+y^12*x^2+y^11*x^3
           +y^6*x^8+y^15+y^14*x+y^5*x^10+y^4*x^11+y^16+y^15*x+y^10*x^6+y^6*x^10
           +y^5*x^11+x^16,
      y^3*x^5+x^8+y^5*x^4+y^3*x^6+y^4*x^6+y^3*x^7+x^10+y^3*x^8+y^9*x^3+y^6*x^6
           +y^4*x^8+y^11*x^2+y^9*x^4+y^13*x+y^9*x^5+y^6*x^8+y^3*x^11+x^14+y^15
           +y^13*x^2+y^9*x^6+y^5*x^10+y^3*x^12+y^14*x^2+y^13*x^3+y^4*x^12
           +y^3*x^13+x^16+y^13*x^4+y^3*x^14+y^14*x^4+y^4*x^14,
```

```
x^6+y^3*x^5+y^5*x^4+x^9+y^6*x^4+y^10*x^2+y^9*x^3+x^12+y^11*x^2+y^6*x^7
     +y^13*x+y^3*x^11+y^15+y^10*x^5+y^5*x^10+x^15}}
```

Indeed it is clear that $C(I^2,R) = C(I,R)^2$ from the leading monomials from the local monomial ordering, but not from the global one given that $C(I,R)$ has x^9, $C(I^2,R)$ has x^{14}, and $C(I^3,R)$ has x^{20}.

2 normalI

For some reason (unknown to me) it took forever (well almost 20000 seconds) using `normalI` in Singular to do this (as compared to fractions of a second in Macaulay2). Fortunately I thought of interchanging variables to test that and it only took 45 seconds! Go figure.

Of import is that `normalI` thinks that there are two levels $C(I,R)$ and $C(I^2,R)$ necessary, since it reports two lists of elements! And the standard basis produced is not minimal. `option(redSB)` is needed to do this.

```
                     SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-5
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   Jul 2012
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "normal.lib";
> ring r=2,(y,x),dp;
> ideal i=y5+x5+y3x+x2,y6+y5x+x6+y5+y4x;
> ideal s=std(i);s;
s[1]=y5+x5+y3x+x2
s[2]=yx5+y3x2+x5+y3x+yx2+x3+x2
s[3]=x10+y4x4+y4x3+x7+y3x3+y2x4+y4x+y2x3+yx3+x4+yx2
> list nori=normalI(i);nori;
[1]:
   _[1]=y5+x5+y3x+x2
   _[2]=y6+y5x+x6+y5+y4x
   _[3]=y8+y4x4+y3x5+y3x4+y6+y4x+y3x
[2]:
   _[1]=x17+y8x7+y8x5+y6x5+x11+y8x+y6x3+y4x5+y2x7+x7
   _[2]=y10+x10+y6x2+x4
   _[3]=y11+y10x+y6x5+x11+y10+y8x2+y5x5+y4x6+y3x7+y8x+y7x2+y6x2+y5x3+x8+y5x2+y4x3
   _[4]=y13+y9x4+y4x9+y3x10+y11x+y8x4+y7x5+y6x6+y3x9+y11+y8x2+y3x7+y8x+y7x2+y4x3+y3x3
   _[5]=y12+y10x2+x12+y10+y8x2
   _[6]=y14+y13x+y10x4+y4x10+y3x11+y13+y12x+y8x5+y7x6+y3x10+y12+y11x+y8x4+y7x5+y6x6+y11+
   _[7]=y16+y8x8+y6x10+y6x8+y12+y8x2+y6x2

> ideal t2=std(nori[2]);t2;
t1[1]=y10+x10+y6x2+x4
t1[2]=y6x5+yx10+y8x2+y5x5+y4x6+y3x7+x10+y8x+y6x3+y5x3+x8+y5x2+y4x3+yx4+x5+x4
```

```
t1[3]=y2x10+y6x4+x10+y6x2+y2x4+x6+x4
t1[4]=y9x4+y4x9+y8x4+y3x9+y8x3+y3x8+yx10+x11+y9x+y8x2+y5x5+y4x6+y5x4+yx8+x9+y5x3+y4x4+x8
t1[5]=x14+y8x4+y4x8+y9x2+y4x7+x11+y8x2+y6x4+y3x7+y2x8+yx9+y8x+y6x3+y5x4+y4x4+y3x5+x8+y3x
t1[6]=y5x9+x14+y9x3+y8x4+y4x8+y3x9+y9x2+y4x7+x11+y9x+y8x2+y6x4+y3x7+y2x8+yx9+y8x+y6x3+y5

> ring r1=2,(y,x),ds;
> ideal i1=imap(r,t1);i1;
i1[1]=x4+y6x2+y10+x10
i1[2]=x4+yx4+x5+y5x2+y4x3+y5x3+x8+y8x+y6x3+y8x2+y5x5+y4x6+y3x7+x10+y6x5+yx10
i1[3]=x4+y2x4+x6+y6x2+y6x4+x10+y2x10
i1[4]=yx4+y3x3+yx5+y5x2+y3x4+y5x3+y4x4+x8+y5x4+yx8+x9+y9x+y8x2+y5x5+y4x6+y8x3+y3x8+yx10+
i1[5]=x5+y3x4+y2x5+y4x4+y3x5+x8+y8x+y6x3+y5x4+y8x2+y6x4+y3x7+y2x8+yx9+y9x2+y4x7+x11+y8x4
i1[6]=x5+y3x3+y3x4+y2x5+y5x3+y4x4+x8+y8x+y6x3+y5x4+y9x+y8x2+y6x4+y3x7+y2x8+yx9+y9x2+y4x7

> ideal j1=std(i1);j1;
j1[1]=x4+y6x2+y10
j1[2]=y3x3+yx5+y5x2+y3x4+y5x3+y4x4+x8+y7x2+y5x4+yx8+x9+y9x
j1[3]=y5x2+y4x3+y6x2+y5x3+x8+y8x+y7x2+y10
j1[4]=y8x+y5x4
j1[5]=y10

> setring r;
> ideal u1=nori[1];u1;
u1[1]=y5+x5+y3x+x2
u1[2]=y6+y5x+x6+y5+y4x
u1[3]=y8+y4x4+y3x5+y3x4+y6+y4x+y3x

> setring r1;
> ideal i2=imap(r,u1);i2;
i2[1]=x2+y3x+y5+x5
i2[2]=y5+y4x+y6+y5x+x6
i2[3]=y3x+y4x+y6+y3x4+y8+y4x4+y3x5

> ideal j2=std(i2);j2;
j2[1]=x2+y3x+y5
j2[2]=y3x
j2[3]=y5+y4x
> timer;
19856
```

Curiously, if one reverses the order on the variables in this example, one gets:

```
 ring r=2,(x,y),dp;
> ideal i=y5+x5+y3x+x2,y6+y5x+x6+y5+y4x;
> list nor=normalI(i);nor;
[1]:
```

```
   _[1]=x5+y5+xy3+x2
   _[2]=x6+xy5+y6+xy4+y5
   _[3]=x4y4+x4y3+x3y3+x2y4+xy5+y5+x4+xy3+x3
> timer;
45
> ideal s=std(nor[1]);s;
s[1]=x5+y5+xy3+x2
s[2]=y6+x2y3+xy4+y5+x3
s[3]=x4y4+x4y3+x3y3+x2y4+xy5+y5+x4+xy3+x3

> ring r1=2,(x,y),ds;
> ideal i1=imap(r,s);
> ideal s1=std(i1);s1;
s1[1]=x2+xy3+x5+y5
s1[2]=xy3+x2y3+y5
s1[3]=y5
```

a much faster result with $x^2(1+y^3+x^3) = s[1]+s[2]+s[3]$, $xy^3(1+x) = s[2]+s[3]$, and $y^5 = s[3]$.

3 Qth-power algorithm

Using my `idealClosure` method in Macaulay2, which uses my Qth-power algorithm to do this, which currently works only over ZZ/q, I get the local answer I want theoretically:

```
R=ZZ/2[y,x,MonomialOrder=>{Weights=>{-1,-1},Weights=>{0,-1}},Global=>false];
i={y^5+x^5+y^3*x+x^2,y^6+y^5*x+x^6+y^5+y^4*x};
wt={{-1,-1},{0,-1}};
time ic=idealClosure(R,{0},i,wt);
     -- used 0.507517 seconds
toString ic
o7 = {x^2, y^3*x, y^5}
```