Units

1 Example 1 with units

How should one deal with units? Consider the example

$$A := \overline{\mathbf{F}}_2[y,x]/\langle y^3x^3 - y^2 - yx - x^2\rangle.$$

The integral closure is

$$C(A, Q(A)) = \overline{\mathbf{F}}_2[u, u^{-1}]/\langle uu^{-1} - 1\rangle$$

with $x = u^3 + u^2 + u$ and $y = u^{-1} + u^{-2} + u^{-3}$.

Let's see how this fares using the various implementations of integral closure algorithms.

2 SINGULAR

```
                    SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-3
                                                            0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \   March 2011
FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> LIB "normal.lib";
> ring r=2,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> int time=timer;
> list nor=normal(i);nor;
//   characteristic : 2
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    T(1) T(2)
//        block   2 : ordering dp
//                  : names    y x
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=yx3+x4+yx+x2
      _[2]=x5+x4+x3+x2
      _[3]=y2+x2
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=y^3*x^3+y^2+y*x+x^2
s[2]=T(2)*x^2+T(2)*x+y^2*x^3+x^5+y+x
s[3]=T(2)*y*x+T(2)*y+y*x^4+x
s[4]=T(2)*y^2+T(2)*x+y^2*x^3+x^4+x^3+x^2+y+x
s[5]=T(1)*x+T(1)+T(2)*y+T(2)+y*x^3+1
s[6]=T(1)*y+T(1)+T(2)*y+T(2)+y*x^3+x^3+x+1
s[7]=T(2)^2+T(1)+x^6+x^4
s[8]=T(1)*T(2)+y*x^4+x^5+y*x^3+x^4+x+1
s[9]=T(1)^2+T(2)*x+T(2)
```

If I computed correctly, $T(2) = u^9 + u^8$ and $T(1) = u^6 + u^4$. But to get u I would need to compute $T(2) + x^3 + x^2 + x$. Why would I think to do this, especially not knowing that the unit u exists?

```
ring r=2,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> int time=timer;
> list norp=normalP(i,"withRing");norp;
//   characteristic : 2
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    T(2)
//        block   2 : ordering dp
//                  : names    y x
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=yx3+x4+yx+x2
      _[2]=x5+x4+x3+x2
      _[3]=y2+x2
> def Rp=norp[1][1];
> setring Rp;
> option(redSB);
> ideal sp=std(norid);sp;
sp[1]=y^3*x^3+y^2+y*x+x^2
sp[2]=T(2)*x^2+T(2)*x+y^2*x^3+x^5+y+x
sp[3]=T(2)*y*x+T(2)*y+y*x^4+x
sp[4]=T(2)*y^2+T(2)*x+y^2*x^3+x^4+x^3+x^2+y+x
sp[5]=T(2)^2*x+T(2)^2+T(2)*y+T(2)+x^7+x^6+x^5+y*x^3+x^4+1
sp[6]=T(2)^2*y+T(2)^2+T(2)*y+T(2)+y*x^6+x^6+y*x^4+y*x^3+x^4+x^3+x+1
sp[7]=T(2)^3+y^2*x^7+x^9+y^2*x^6+x^8+x+1
```

This is the same with $T(1) = T(2)^2 + x^6 + x^4$ eliminated.

3 Macaulay2

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : A=ZZ/2[y,x]/(y^3*x^3-y^2-y*x-x^2);

i2 : time icf=icFractions A;
     -- used 0.158133 seconds

i3 : toString icf

o3 = {(y^2*x^3+y*x^4+x^4+x^3+x^2+y)
     /(y^2+y*x+y+x),
       y,
       x}

i4 : time icp=icFracP A;
     -- used 0.0635558 seconds

i5 : toString icp

o5 = {y,
     (y*x^2+y)
    /(y+x),
     (y*x^2+y*x+x+1)
    /(y^2+1)}

i6 : G=transpose gens gb presentation integralClosure A

o6 = {-6} | y3x3+y2+yx+x2                                                      |
     {-5} | w_(1,1)x2+w_(1,1)x+y2x3+x5+y+x                                     |
     {-5} | w_(1,1)yx+w_(1,1)y+yx4+x                                           |
     {-5} | w_(1,1)y2+w_(1,1)x+y2x3+x4+x3+x2+y+x                               |
     {-7} | w_(1,1)^2x+w_(1,1)^2+w_(1,1)y+w_(1,1)+x7+x6+x5+yx3+x4+1            |
     {-7} | w_(1,1)^2y+w_(1,1)^2+w_(1,1)y+w_(1,1)+yx6+x6+yx4+yx3+x4+x3+x+1 |
     {-9} | w_(1,1)^3+y2x7+x9+y2x6+x8+x+1                                      |
```

Again even knowing that $(y^2x^3 + yx^4 + x^4 + x^3 + x^2 + y)/(y^2 + yx + y + x)$ should be $u^9 + u^8$, what is u? And since $y(x + 1)^2/(y + x)$ should be $1 + u^2$ and $(yx^2 + yx + x + 1)/(y^2 + 1)$ should be $u^{-2} + u^{-1}$, what is u?

4 Magma

```
F:=GF(2);
P<y,x>:=PolynomialRing(F,2);
I:=ideal<P|y^3*x^3-y^2-y*x-x^2>;
N:=Normalisation(I);
y@N[1][2];
x@N[1][2];
G:=GroebnerBasis(N[1][1]);G;

$.1^3 + $.1^2*$.2 + $.1*$.2
$.1^2*$.2 + $.1*$.2 + $.1
[
    $.1*$.2 + $.2^2 + $.2 + 1
]
```

Here $S.2 = u$ and $S.1 = u + 1 + u^{-1}$, so this implementation does really well.

```
FF<x>:=FunctionField(F);
PP<y>:=PolynomialRing(FF);
f:=y^3*x^3-y^2-y*x-x^2;
RER<Y>:=RationalExtensionRepresentation(FunctionField(f));
C<X>:=CoefficientRing(RER);
INT:=Integers(C);
IC:=IntegralClosure(INT,RER);
B:=Basis(IC);B;
[ 1, X^3/(X + 1)*Y + 1/(X + 1), X^2/(X^2 + 1)*Y^2 + (X^2 + 1)/X*Y + 1/(X^2 + 1)
]
```

But this doesn't! $B[2]$ seems to be $u^5 + u^4$ while $B[3]$ seems to be $u^2 + 1$.

5 Integral extensions

Had I used $z := yx^3$ to get a related integral extension problem:

$$\overline{\mathbf{F}}_2[z, x]/\langle z^3 + z^2 + zx^4 + x^8 \rangle$$

I could have used $s^4 := z$ and $u := x/s$ to get the required integral closure and parameterization. But, in general, non-linear transformations to produce integral extensions don't necessarily give the whole integral closure when mapped back to the original problem.

6 char not 2

```
> ring r=0,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> list nor=normal(i);nor;
//   characteristic : 0
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    T(1)
//        block   2 : ordering dp
//                  : names    y x
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=y
      _[2]=x
> def R=nor[1][1];
> setring R;
> option(redSB);
> ideal s=std(norid);s;
s[1]=y^3*x^3-y^2-y*x-x^2
s[2]=T(1)*x-y
s[3]=T(1)*y-y^3*x^2+y+x
s[4]=T(1)^2+T(1)-y^3*x+1

> ring r=23,(y,x),dp;
> ideal i=y3x3-y2-yx-x2;
> list norp=normalP(i,"withRing");norp;
//   characteristic : 23
//   number of vars : 2
//        block   1 : ordering dp
//                  : names    T(1)
//        block   2 : ordering dp
//                  : names    x
//        block   3 : ordering C
[2]:
   [1]:
      _[1]=y
      _[2]=x
> def Rp=norp[1][1];
> setring Rp;
> option(redSB);
> ideal sp=std(norid);sp;
sp[1]=T(1)^3*x^4-T(1)^2-T(1)-1
```

Does it really make sense to have $T(1) := y/x$ integral in terms of y and x in the first? Well, the second shows what one gets by replacing y by $T(1)x$. The result is not really integral any more. Yet it is theoretically correct. Hmm....

Well, if we treat $u^{-1} := T(1)$ as a unit, and work over

$$\overline{\mathbf{F}}\{u\} := \overline{\mathbf{F}}[u, u^{-1}]/\langle uu^{-1} - 1\rangle$$

then

$$C(A, Q(A)) = \overline{\mathbf{F}}\{u\}[x]/\langle x^4 - u^3 - u^2 - u\rangle$$

for characteristic $q \neq 2$.

7 Example 2 with units

Consider something as simple as the Klein quartic:

$$A := \overline{\mathbf{F}}_2[y,x]/\langle y^3 + x^3y + x\rangle$$

in its (standard) two point form in that the divisors are

$$((y)) = -3 \cdot P + 2 \cdot Q + 1 \cdot R$$

and

$$((x)) = -2 \cdot P - 1 \cdot Q + 3 \cdot R$$

with two points at which there are poles.

This is integrally closed. But it contains the unit $u := y^3/x = 1 + yx^2$, with $u^{-1} = 1 + yx^2 + y^2x^4 + x^7$. So perhaps this should be viewed as

$$A = C(A, Q(A)) = \overline{\mathbf{F}}_2\{u\}[x]/\langle x^7 + u^2 + u + 1 + u^{-1}\rangle.$$

If so, then this raises the question of how one detects the existence of units in a problem, since it was only clear to me in this problem from the divisors,

$$((u)) = -7 \cdot P + 7 \cdot Q$$

and hence

$$((u^{-1})) = 7 \cdot P - 7 \cdot Q$$

information not usually used in computing an integral closure.