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Articles

Complementarity in a simple general equilibrium production model

HENRY THOMPSON University of Tennessee

Abstract. A simple three-factor, two-good general equilibrium model, allowing complementarity in production, is examined. Conclusions of general equilibrium economics with two factors must be reinterpreted and qualified when complementarity is possible. Sign patterns of how changing factor endowments affect outputs and how changing prices affect factor payments are uncovered for a small open economy. Results should prove valuable to a wide range of economists interested in extending general equilibrium analysis to include complementary factors of production.

La complémentarité dans un modèle simple d'équilibre général de la production. L'auteur examine un modèle d'équilibre général (deux biens, trois facteurs de production) qui postule qu'il y a complémentarité dans le processus de production. Les conclusions des analyses conventionnelles d'équilibre général avec deux facteurs de production doivent être réinterprétées et qualifiées quand la complémentarité est possible. On découvre des patterns de signes indiquant comment les dotations en facteurs affectent les niveaux de production et comment des changements de prix affectent la rémunération des facteurs de production dans une petite économie ouverte. Ces résultats vont s'avérer utile pour toute une gamme d'économistes intéressés à étendre l'analyse d'équilibre général aux cas où les facteurs de production sont complémentaires.

INTRODUCTION

Attention has recently turned to developing the simplest general equilibrium model allowing complementarity in production, the three-factor, two-good model. Explicit treatment of natural resources or skilled labour in addition to labour and capital increases the model's range of application. Sign patterns of effects of exogenous endowment changes upon outputs and exogenous price changes upon factor payments for a small open economy are derived in this paper. Results should be of interest in any field of economics employing general equilibrium models: international trade, public finance, industrial organization, and others.

Factor price equalization no longer holds when a factor is added to the two-factor, two-good model, as pointed out by Samuelson (1953-54). Three factors simultaneously complicate the factor intensity ordering. A 'middle' factor is created, as the two other factors in the ordering become 'extreme,' using terminology introduced by Ruffin (1981). An important question concerns robustness of Rybczynski (1955) or symmetric Stolper-Samuelson (1941) results. A strong Rybczynski result, for instance, holds in the two-factor model. Each good's output, at constant prices, is postively correlated with the endowment of the factor used intensively in that industry, negatively with the other.

Batra and Casas (1976) claim a strong Rybczynski result stated in terms of extreme factors is also found in the three-factor model. Suzuki (1983) and Jones and Easton (1983) point out, as is done here, that a strong Rybczynski result is not necessary.[1] Takayama (1982) develops general properties of the three-factor model, investigating special cases of the Rybczynski result. Stolper-Samuelson-Rybczynski partial derivative sign patterns are derived in the present paper. An increase in the endowment of an extreme factor may lead to an increase in the output of both industries. Perhaps more surprisingly, there may occur a decrease in the output of the industry where the factor is extreme.

Complementarity between two factors, where a rising payment to one causes a reduction in the other's usage, contributes to these possibilities. As sterile as these results may seem, they lie at the core of much reasoning in general equilibrium economics. A good deal of intuition is based on the simpler two-factor situation, where complementarity is impossible. A theorem is introduced which leads to derivation of Stolper-Samuelson-Rybcznyski sign patterns. Results are discussed in an effort to develop intuition for this useful general equilibrium model.

MODEL

Jones and Scheinkman (1977) and Chang (1979) summarize properties of general equilibrium production models. Let v_i and p_j ($i = 1, 2, 3$, and $j = 1, 2$) represent exogenous factor endowments and prices, while w_i and x_j represent

Special thanks go to Roy Ruffin and Joel Sailors. The core of this paper was completed under their guidance, and served as part of the author's 1981 doctoral dissertation at the University of Houston. Thanks are also due to Ron Jones and Akira Takayama for thoughtful remarks. Betty Wickham rendered a thorough critique of an earlier draft.

1. This conclusion was reached concurrently and independently by Suzuki and the author.

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endogenous factor payments and outputs. Production functions exhibit constant returns to scale. An exogenous price or factor endowment change causes factor payments and outputs to adjust in a manner consistent with full employment and competitive pricing. Firms determine cost minimizing, unit factor mixes a_{ij} according to the vector of factor payments. A particular solution of factor mixes is represented by a 3×2 technology matrix A. Factors are renumbered if necessary, so $a_{11}/a_{12} > a_{21}/a_{22} > a_{31}/a_{32}$. Factor 1 (3) is extreme in industry 1 (2), while factor 2 is the middle factor. Goods are rescaled so $a_{11} = a_{22} = 1$. For notational purposes, $b_1 \equiv 1 - a_{12}a_{21}$, $b_2 \equiv a_{21}a_{32} - a_{31}$, $b_3 \equiv a_{32} - a_{12}a_{31}$, $c_1 \equiv b_2 + b_3$, $c_2 \equiv b_1 - b_2$, and $c_3 \equiv b_1 + b_3$. Given the factor extremity condition, b_1, b_2, b_3, c_1 and c_3 are positive, with the sign of c_2 depending upon factor mix values.

Aggregate substitution between factors h and k is expressed by the substitution term $s_{kh} = \Sigma_j x_j \, \partial a_{kj}/\partial w_h$. The 3×3 matrix of substitution terms is symmetric and negative semidefinite. A result of cost minimizing behavior is $\Sigma_i s_{hi} w_i = 0$, for every factor h. Rescale factors so the payment to each equals one, or $\Sigma_i s_{ih} = 0$. Summing rows in the substitution matrix yields zero.

$$s_{11} + s_{12} + s_{13} = s_{12} + s_{22} + s_{23} = s_{13} + s_{23} + s_{33} = 0. \tag{1}$$

Summarizing the general equilibrium model,

$$\begin{bmatrix} S & A \\ A' & 0 \end{bmatrix} \begin{bmatrix} dw \\ dx \end{bmatrix} = \begin{bmatrix} dv \\ dp \end{bmatrix}. \tag{2}$$

Cramer's rule is used to solve comparative statics. Chang (1979) shows the sign of the system determinant D to be $(-1)^3$ with three factors. A symmetric, positive semidefinite matrix of $\partial x/\partial p$ terms describes the production possibility frontier. Each industry expands in response to a rising price for its output ($0 < \partial x_j/\partial p_j$, $j = 1, 2$,), while the other industry contracts with endowments unchanged ($0 > \partial x_m/\partial p_q$, $m \neq q$).

Samuelson's well known reciprocity holds due to symmetry. Where $w_{kh} \equiv \partial w_k/\partial v_h$, it follows that $w_{kh} = w_{hk}$. Diagonal elements are found by dividing (2) by dv_h, then solving for w_{hh}. Thus $w_{11} = b_2^2/D$, $w_{22} = b_3^2/D$, and $w_{33} = b_1^2/D$. Each factor's general equilibrium demand curve is downward sloping, as indicated in the literature.

Ruffin (1981) uses a diagrammatic argument to conclude extreme factors are 'enemies,' an increase in the endowment of one lowering the other's payment. Both extreme factors are a 'friend' of the middle factor. These terms can be directly signed: $w_{12} = -b_2b_3/D > 0$, $w_{13} = b_1b_2/D < 0$, and $w_{23} = -b_1b_3/D > 0$. Whether two factors are friends is completely independent of technical substitution in any model containing one more factor than good. Thompson and Clark (1983) apply this model to the U.S. economy, showing capital and skilled labour are enemies, while each has labour as a friend.

RYBCZYNSKI SIGN PATTERNS

Samuelson's reciprocity also indicates $r_{mk} = q_{km}$, where $r_m \equiv \partial x_m/\partial v_k$ and $q_{km} \equiv \partial w_k/\partial p_m$. Consider the Rybczynski matrix,

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}.$$

Sign patterns are of interest, as in the above analysis of $\partial x/\partial p$ and $\partial w/\partial v$ matrices.

It is known that $\Sigma_i a_{i1} r_{2i} = \Sigma_i a_{i2} r_{1i} = 0$. Each row of R must thus have at least one positive and one negative element. Increasing the endowment of a factor shifts the production possibility curve outward everywhere, so an increase in at least one output results. Every column of R must thus have at least one positive element. These results may be combined to yield twelve sign patterns.

Increases in the endowment of factor k lower the output of good m where $r_{mk} < 0$. Factor k can then be said to "weaken" industry m. If $r_{mk} > 0$, factor k 'strengthens' the industry. A proof of the following theorem is presented in the appendix.

THEOREM. *If an industry is weakened by its extreme factor, then both other factors cannot strengthen that industry.*

Remaining Rybcznyski sign patterns are:

$$\begin{matrix} + & + & - \\ - & + & + \\ & \text{(a)} & \end{matrix} \qquad \begin{matrix} + & - & - \\ - & + & + \\ & \text{(b)} & \end{matrix} \qquad \begin{matrix} + & - & + \\ - & + & + \\ & \text{(c)} & \end{matrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ & \text{(d)} & \end{matrix}.$$

Switching names of extreme factors and then goods gives rise to three additional patterns, structurally the same as (b), (c), and (d). Jones and Easton (1983) consider solutions to the system with prices changing and payment to the middle factor constant, uncovering Stolper-Samuelson magnification effects.

RESULTS

Pattern (a) results from the specific factors model, where an increase in the common factor strengthens both industries. Only (a) and (b) exhibit a strong Rybczynski pattern. It can be seen that any factor may strengthen both industries. Either or both industries may be strenthened by two factors; either, but not both, may be weakened by two factors. Pattern (d) may be the most surprising, in that an industry is weakened by its extreme factor. If this is so, the industry is also weakened by the other extreme factor. In the Stolper-Samuelson matrix, this means that a tariff may lead to a decreasing payment to the factor extreme in the protected industry.

Insight can be gained by postulating conditions for case (d). Suppose factor 3 is used very intensively in industry 2, to the extent its marginal product is rapidly declining. Let an exogenous increase in the endowment of factor 3 occur. Marginal productivity of factor 3 declines faster in sector 2 than in 1, so additional factor 3 is attracted to industry 1. Factors 2 and 3 are combined more productively in industry 1, where they are strong complements. As w_3 falls, the demand for factor 2 in industry 1 rises, causing its wage in that sector to rise. Factor 2 migrates to sector 1 until wages are again equal across industries. Owners of factors 1 and 3 suffer lower payments, with the payment to factor 2 rising. In the process, x_2 declines as x_1 increases.

A comprehensive treatment of production adjustments with complementarity present is beyond the scope of this paper. Each comparative static sign pattern could result from various factor mix and substitution situations. Importance of the three-factor model in a number of fields of economics will grow when its structure has been more completely developed and intuitively grasped. Analysis of comparative static sign patterns should prove valuable in this process.

APPENDIX

A proof of the theorem in the paper follows. It is claimed pattern $(- + +)$ in the top row of R is impossible. Suppose this pattern holds. Solve (2) for the first row of Rybczynski terms. Let R_{mk} represent the cofactor of r_{mk}; that is, $R_{mk} \equiv Dr_{mk}$. Note R_{mk} and r_{mk} have opposite signs. Using (1), where $d_1 \equiv c_1 s_{12}$, $d_2 \equiv c_2 s_{13}$, and $d_3 \equiv c_3 s_{23}$, it is straightforward to show

$$\begin{aligned} R_{11} &= -a_{32}d_1 - d_2 - (1 + a_{32})d_3 \\ R_{12} &= -a_{32}d_1 + (a_{12} + a_{32})d_2 + a_{12}d_3 \\ R_{13} &= (1 + a_{12})d_1 - d_2 + a_{12}d_3. \end{aligned} \tag{3}$$

Negative semidefiniteness of the substitution matrix implies a positive principle minor, $s' \equiv s_{11}s_{22} - s_{12}^2$. It follows from (1) that $s_{23} > s_{12}s_{13}/s_{11}$. Substitute this inequality into (3), where $N_1 \equiv (d_1 - d_2)(a_{32}s_{12} - s_{13})$, $N_2 \equiv (d_1 - d_2)(a_{32}s_{12} + (a_{12} + a_{32})s_{13})$, and $N_3 \equiv (d_1 - d_2)[-(1 + a_{12})s_{12} - s_{13}]$. Thus $0 < R_{11} < N_1/s_{11}$, $0 > R_{12} > N_2/s_{11}$, and $0 > R_{13} > N_3/s_{11}$. Signs of these numerators are determined, since $s_{11} < 0$: $N_1 > 0$, $N_2 < 0$, and $N_3 < 0$. Simplifying first the inequality $N_1 > N_2$, then $N_1 > N_3$,

$$d_1 s_{13} > d_2 s_{13} \tag{4}$$

$$d_1 s_{12} < d_2 s_{12}. \tag{5}$$

There are three possible sign patterns for the pair of substitution terms (s_{12}, s_{13}), namely $(+ +)$, $(+ -)$, and $(- +)$. In the first the contradiction that d_1 is both greater and less than d_2 follows from (4) and (5). In the second case, where s_{13} is negative, equation (5) implies $d_2 > 0$, since $d_1 s_{12} > 0$. By (3), $R_{11} < 0$, so pattern 1 cannot hold.

Once again $d_2 > 0$ in the last case, where $s_{12} < 0$. From expressions for R_{11} and R_{13} in (3), it follows that (i) $-d_1 - d_2 > 2d_3$; and (ii) $-2d_1 > -d_2 + d_3$. Since $c_2 = c_3 - c_1$, it is seen that (iii) $c_1(s_{13} - s_{12}) > c_3(2s_{23} + s_{13})$; and (iv) $c_1(2s_{12} + s_{13}) < c_3(s_{13} - s_{23})$. With s_{12} negative, $s_{13} - s_{12} > 0$. Thus from (iii), $c_1 > c_3(2s_{23} + s_{13})/(s_{13} - s_{12})$. Substituting this into (iv), and multiplying both sides of the result by $(s_{13} - s_{12})$, another contradiction, $s' < 0$, follows.

A proof that $(- + +)$ is impossible in the top row of the Rybczynski matrix is completed. Sign pattern $(+ + -)$ in the bottom row is impossible as well, since names of extreme factors and goods are arbitrary.

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