General Equilibrium Production with Constant Elasticity of Substitution

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Properties of general equilibrium models with constant elasticity of substitution CES production may be underappreciated. Comparative static price elasticities are generally insensitive to the degree of CES substitution, a surprise with substitution ranging from nearly zero to nearly perfect. The influence of the degree of CES substitution is examined and properties are illustrated in an application to the United Arab Emirates.

Comparative static elasticities in competitive general equilibrium models of production generally depend on the degree of substitution as well as factor intensities or factor shares. In models with the same number of factors and products, Stolper-Samuelson (1941) elasticities of prices on factor prices are known to be independent of substitution.

The present note shows that in models with more factors than products, including specific factors models, these general equilibrium elasticities are independent of the degree of constant elasticity of substitution. Output adjustments absorb the effects of substitution leaving factor price effects unchanged, and similar results hold for Rybczynski endowment elasticities of outputs. A higher degree of substitution does, however, lower endowment elasticities of factor prices and price elasticities of outputs.

These properties are illustrated in a model calibrated for the United Arab Emirates economy with four major sectors and six types of labor along with capital input.

1. CES Substitution in Competitive General Equilibrium Production

Substitution elasticities describe the adjustment potential in cost minimizing inputs with respect to factor prices. The cross price elasticity between the price of factor *k* and the input of factor *i* in product *j* is

$$E_{ij}^{k} = \hat{a}_{ij}/\hat{w}_{k} = \theta_{kj} S_{ij}^{k} \tag{1}$$

where hats ^ represent percentage changes, a_{ij} is the cost minimizing input of factor *i* in product *j*, w_k is the price of factor *k*, θ_{kj} is the share of factor *k* in industry *j* revenue, and S_{ij}^{k} is the Allen (1938) partial elasticity of substitution from the production function. With Cobb-Douglas production $S_{ij}^{k} = 1$ and the cross price elasticity equals the *k* factor share. Linear homogeneity implies $\sum_k E_{ij}^{k} = 0$ and the own price elasticity E_{ij}^{i} is derived in practice as the negative of the sum of the cross price elasticities.

The homogeneous CES production function introduced by Arrow, Chenery, Minhas, and Solow (1961)

$$x = (a_1 v_1^{\rho} + a_2 v_2^{\rho})^{1/\rho} \tag{2}$$

has output is produced by inputs v_i, $i = 1, 2$. The a_i are constant positive distribution parameters and the substitution parameter ρ must lie in the range $0 < \rho < 1$ for diminishing returns. CES generalizes Cobb-Douglas to any degree of the Allen partial elasticity of substitution, converging to Cobb-Douglas in the limit as $\rho \to 0$ with constant returns to scale (CRS). To see this property take the log of (2) $\ln x = \ln(a_1 K^{\rho} + a_2 L^{\rho})/\rho$ and use l'Hospital's rule as $\rho \to 0$ to find $\ln x = [d(a_1 v_1^{\rho} + a_2 v_2^{\rho})/d\rho]/(a_1 v_1^{\rho} + a_2 v_2^{\rho}) = (a_1 v_1^{\rho} \ln v_1 + a_2 v_2^{\rho} \ln v_2)/(a_1 v_1^{\rho} + a_2 v_2^{\rho}) = a_1 \ln v_1 + a_2 \ln v_2$ where $a_1 + a_2 = 1$ with CRS.

Chung (1994) shows CES Allen elasticities S_{ij}^{k} are constant, independent of factor prices, and identical for every pair of factors as suggested by Uzawa (1962). CES generalizes Cobb-Douglas in the strong sense that CES Allen elasticities are independent of input levels and identical for every pair of factors. The degree of CES is denoted by $\alpha > 0$ and the Allen and cross price elasticities E_{ij}^{i} in (1) are scaled by α.

Aggregate substitution elasticities σ_{ik} for the economy are a weighted average of the cross price elasticities for each sector as described by Takayama (1982) and Thompson (1994),

$$\sigma_{ik} \equiv \hat{a}_i/\hat{w}_k = \Sigma_j \lambda_{ij} E_{ij}^{k} = \Sigma_j \lambda_{ij} \theta_{kj} S_{ij}^{k}, \tag{3}$$

with factor shares θ_{kj} and industry shares λ_{ij} of factor i in industry j in the derivation of aggregate substitution elasticities.

Full employment and competitive pricing are stated $\Sigma_j a_{ij} x_j = v_i$ and $\Sigma_i a_{ij} w_i = p_j$ where x_j is the output of good j, v_i is the endowment of factor k, w_i is the price of factor i, and p_j is the price of good j. Differentiating the two conditions leads to

$$\Sigma_i \sigma_{ki} \hat{w}_i + \lambda_{kj} \hat{x}_j = \hat{v}_k \text{ and} \tag{4}$$

$$\Sigma_i \theta_{im} \hat{w}_i = \hat{p}_m \tag{5}$$

for input k and output m where (5) simplifies with the cost minimizing envelope property.

The competitive general equilibrium model from (4) and (5) with exogenous factor endowments and prices in matrix form is

$$\begin{pmatrix} \sigma & \lambda \\ \theta' & 0 \end{pmatrix} \begin{bmatrix} \hat{w} \\ \hat{x} \end{bmatrix} = \begin{bmatrix} \hat{v} \\ \hat{p} \end{bmatrix} \tag{6}$$

where σ is the matrix of substitution elasticities, λ the matrix of industry shares, θ' the transposed matrix of factor shares, and *0* a null matrix.

2. Solution of the CES Model

The model relates exogenous comparative static changes in prices and endowments to endogenous adjustments in factor prices and outputs given full employment and competitive pricing. Comparative static partial elasticities are found by inverting (6). For reference,

$$F = \hat{w}/\hat{v} \qquad S = \hat{w}/\hat{p} \qquad R = \hat{x}/\hat{v} \qquad P = \hat{x}/\hat{p}\,. \tag{7}$$

The economy wide cross price elasticities of CES substitution in matrix σ of (6) are

$$\sigma_{ik} = \alpha\Sigma_j\lambda_{ij}\theta_{kj}, \tag{8}$$

where α is the Allen elasticity of substitution. Letting σ_{CD} represent the matrix of Cobb-Douglas substitution elasticities derived with unit Allen elasticities, the CES substitution matrix is $\alpha\sigma_{CD}$. Multiply the CES system matrix by its inverse to find the identity matrix

$$\begin{pmatrix} \alpha\sigma_{CD} & \lambda \\ \theta' & 0 \end{pmatrix} \begin{pmatrix} F & S \\ R & P \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9}$$

implying

$$\alpha\sigma_{CD}F + \lambda R = 1 \tag{10}$$

$$\alpha\sigma_{CD}S + \lambda P = 0 \tag{11}$$

$$\theta' F = 0 \tag{12}$$

$$\theta' S = 1. \tag{13}$$

Although the CES scalar α does not appear in (12) or (13) it may affect F and S in the solution of the full set of equations depending on model dimensionality.

3. CES Substitution in Even Models

In "even" models with the same number of factors and products, the trivial solution to the homogenous $\theta' F = 0$ in (10) is the null F matrix. The factor price equalization FPE result is that endowment differences do not affect factor prices between economies with identical prices due to free trade. Deriving cofactors of the σ terms in (6) to solve for elements of F, the null matrix dominates the cofactors leading to the FPE property $\hat{w}/\hat{v} = 0$.

Pre-multiply $\theta' S = 1$ in (10) by θ'^{-1} to find another familiar property of even models, that the Stolper-Samuelson matrix S is the inverse of θ'. Cofactors of $\hat{w}/\hat{p}$ terms contain the determinant $\det\lambda$ but that is cancelled by the determinant $\Delta = \det\lambda \det\theta$. Stolper-Samuelson results do not depend on substitution when S and θ' in (10) are square.

When $F = 0$ the implication of (10) is that $R = \lambda^{-1}$ in the familiar Rybczynski reciprocity property. Cofactors of $\hat{x}/\hat{v}$ terms in (6) contain $\det\theta$ but it is cancelled by the determinant. The effects of endowments on outputs are independent of substitution in even models.

With the Stolper-Samuelson terms in S unaffected by the degree of substitution α it follows from (11) that P must change by a factor of α. Cofactors of the $\hat{x}/\hat{p}$ terms in (6) contain the determinant $\det\sigma$ and they change by α. The production frontier becomes less concave with increased CES substitution implying magnified output effects of price changes.

Figure 1 illustrates the adjustment process in the 2x2 model. Industry 1 uses factor 1 intensively and an increase in the relative price p_1/p_2 raises the relative price of factor 1 from w_2/w_1 to w_2'/w_1'. The factor price adjustment is identical regardless of substitution while output adjustments depend on substitution. With homothetic production there is less of a decline in the input ratio v_{ij}/v_{2j} and less output adjustment with a higher degree of substitution. A lower degree of substitution results in more of increase in x_1 and more of a decrease in x_2.

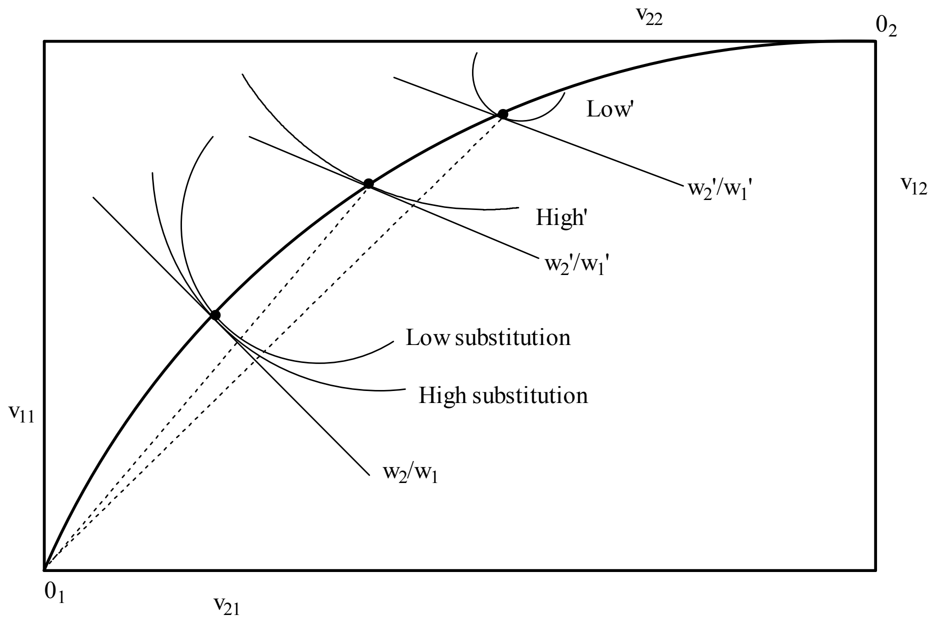

Figure 1. Stolper-Samuelson Price Adjustment with CES

4. CES Substitution in Uneven Models

In uneven models with more factors than products, including specific factors models, there are nontrivial solutions to the homogenous system (12) with nonzero terms in the F matrix implying factor price equalization does not hold.

In the solution to (6) the null matrix 0 has lower dimensions than the substitution matrix σ implying nonzero cofactors for the σ matrix terms. For given Rybczynski terms R, factor price elasticities in F uniformly scale down by α according to $\alpha\sigma_{CD}F + \lambda R = 1$ in (10). Rybczynski terms in R are insensitive to the degree of CES substitution in α.

The 3x2 model of Ruffin (1981), Takayama (1982), and Thompson (1985) has a σ matrix with one more column than λ and one more row than θ'. Its system determinant Δ contains a linear component of substitution terms $\alpha\sigma_{ik}$ and the degree of CES substitution α can be factored out. See the literature for the complete derivation.

As apparent from (9) cofactors of the *F* matrix in the 3x2 model are corresponding terms in the σ matrix, eliminating a column and row of the substitution matrix σ. As a result, cofactors of *F* contain no substitution terms. Increasing the degree of CES substitution α then reduces elements of *F* accordingly as the cofactors of *F* are divided by the determinant Δ.

Cofactors of the θ' matrix in the 3x2 Stolper-Samuelson matrix *S* in (6) contain a linear component of substitution terms $\alpha\sigma_{ik}$ with rows or columns of the substitution matrix in each cofactor. This linear dependence on α matches that in the determinant, implying the degree of CES substitution has no effect on *S*. Rybczynski terms in *R* are similar in their linear component of $\alpha\sigma_{ik}$ terms matching that in Δ making Rybcynski terms insensitive to the degree of CES substitution α in the 3x2 model. The same property applies to any model with one more factor than product including specific factors models.

With *S* not affected by CES substitution in the 3x2 model (11) implies concavity in the production frontier *P* matrix decreases with the degree of CES substitution. Cofactors in *P* include quadratic substitution terms since the σ matrix has two more rows or columns than λ or θ' in the cofactors. The $\hat{x} / \hat{p}$ terms scale by α in any model with one more factor than product.

The adjustment process is illustrated in the specific factors model in Figure 2 focusing on a single sector with specific factor 2 and shared factor 1, and unit value isoquants $x_j = 1/p_j$. The input ratio is v_1/v_2 in the original equilibrium with either High or Low CES. A higher price shifts these unit value isoquants toward the origin leading to an increase in the relative price of factor 2 from w_2/w_1 to w_2'/w_1' with High or Low CES. High CES leads to more of an increase in the input ratio v_1/v_2. Given a fixed endowment of the specific factor, output x_2 increases more with a higher degree of substitution.

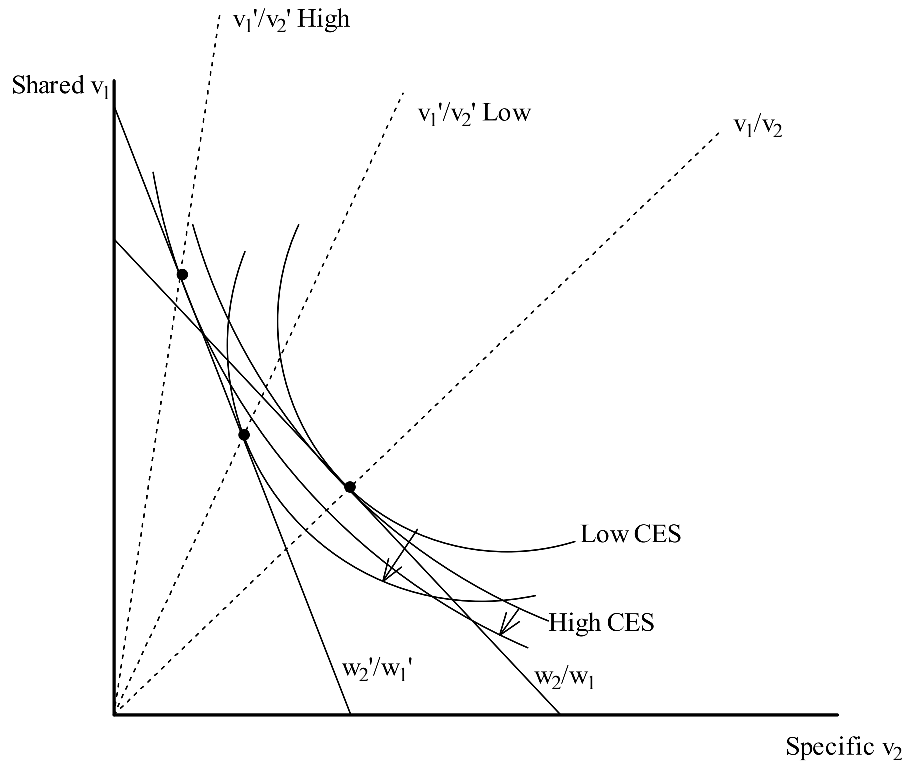

Figure 2. Sector Adjustment in the Specific Factors Model with CES

Models with four or more factors raise the power of the effect of α on substitution terms in both cofactors and the determinant implying the same sensitivity to α as in the 3x2 model. The present properties generalize to any number of factors beyond the number of products, including specific factors models with multiple specific inputs.

When there are more products than factors, the problem of indeterminacy arises. In the 2x3 model, the 3x3 null matrix in (9) implies a zero determinant. Assumptions such as maximized utility and balanced trade can close the model but are beyond the scope of the present pure production models.

5. Prices and Factor Prices in the UAE

A calibration of the United Arab Emirates economy illustrates these properties of CES substitution. Table 1 presents the factor share matrix θ derived directly from national income accounts for capital K and six types of labor

L_G	Managers	L_P	Professional
L_C	Clerks	L_L	Sales
L_R	Agricultural	L_O	Operators

across four sectors

x_A	Agriculture	x_I	Industry
x_M	Manufactures	x_S	Services.

The source of the data is the Dubai Municipality Statistics Center (2003). Capital shares are derived as residuals of industry value added after labor payments. The share of value added paid to capital is overstated because data on energy and natural resources is not available. Capital can be considered an aggregate of nonlabor inputs. Table 2 reports industry shares derived assuming equal prices for factors across industries.

Table 1. Factor Shares θ_{ij}

	x_A	x_I	x_M	x_S
L_G	0.03	0.02	0.03	0.05
L_P	0.10	0.04	0.06	0.10
L_C	0.03	0.00	0.01	0.03
L_L	0.02	0.00	0.01	0.09
L_R	0.07	0.00	0.00	0.00
L_O	0.02	0.00	0.01	0.00
K	0.72	0.93	0.89	0.72

Table 2. Industry Shares λ_{ij}

	x_A	x_I	x_M	x_S
L_G	0.05	0.35	0.19	0.41
L_P	0.08	0.33	0.20	0.39
L_C	0.11	0.18	0.14	0.57
L_L	0.04	0.05	0.09	0.83
L_R	0.99	0.00	0.01	0.00
L_O	0.28	0.19	0.31	0.22
K	0.04	0.54	0.22	0.21

Agricultural labor L_R is nearly specific to Agriculture x_A. Gauging by factor shares alone, industry x_I uses capital intensively as well as labor inputs of Managers L_M and Professionals L_P. Services output x_S intensively uses Professionals L_P but also Clerks L_C and Sales L_L, and has the largest industry share of all types of labor except Agricultural L_R and Operators L_O.

Table 3 shows substitution elasticities for CES = 1 (Cobb-Douglas) production. Labor inputs are relatively sensitive to the price of capital while capital input is insensitive to wages. Labor inputs have nearly unit own elasticities and capital a much weaker own elasticity.

A higher degree of CES substitution scales substitution elasticities accordingly. The vast literature on applied production analysis typically reports weak or inelastic substitution as reviewed by Thompson (2006a, 2006b). Some recent examples include Vincent, Lang, Seok (1992), Chang (1994), Kemfert (1998), Mahmud (2000), Yi (2000), Balistreri, McDaniel, Wong (2002), and Urga and Walters (2003). Models with CES parameters α = 1 and 0.1 gauge sensitivity in the following calibrations.

Table 3. Substitution Elasticities, CES = 1

	$\mathbf{w_G}$	$\mathbf{w_P}$	$\mathbf{w_C}$	$\mathbf{w_L}$	$\mathbf{w_R}$	$\mathbf{w_O}$	$\mathbf{w_K}$
$\mathbf{a_G}$	-0.97	0.07	0.02	0.04	0.00	0.00	0.83
$\mathbf{a_P}$	0.03	-0.93	0.02	0.04	0.01	0.01	0.82
$\mathbf{a_C}$	0.04	0.08	-0.98	0.06	0.01	0.01	0.78
$\mathbf{a_L}$	0.05	0.09	0.03	-0.92	0.00	0.00	0.75
$\mathbf{a_R}$	0.03	0.10	0.03	0.02	-0.93	0.02	0.72
$\mathbf{a_O}$	0.03	0.08	0.02	0.03	0.02	-0.99	0.81
$\mathbf{a_K}$	0.03	0.06	0.01	0.02	0.00	0.00	-0.13

Tables 4 and 5 present the comparative static elasticities. Stolper-Samuelson $\hat{w}/\hat{p}$ and Rybczynski $\hat{x}/\hat{v}$ elasticities are identical for CES = 1 and 0.1. The strongest positive $\hat{w}/\hat{p}$ effects occur for larger factor shares: between agricultural wages w_R and prices p_A, the capital return w_K and industry prices p_I, professional w_P and operator wages w_O and manufacturing prices p_M, and sales wages w_L and service prices p_S.

Table 4. Comparative Static Results CES = 1

	v_G	v_P	v_C	v_L	v_R	v_O	v_K		p_A	p_I	p_M	p_S
w_G	-0.89	0.21	0.09	0.17	-0.02	0.05	0.39		-0.26	-5.49	5.27	1.38
w_P	0.13	-0.74	0.11	0.10	-0.01	0.12	0.28		-0.13	-11.1	12.2	-0.16
w_C	0.15	0.32	-0.86	0.38	0.05	0.01	-0.04		0.66	-3.57	-0.11	4.04
w_L	0.19	0.38	0.17	-0.27	-0.00	-0.11	-0.35		-0.02	3.23	-11.3	9.36
w_R	-0.16	0.25	0.09	-0.07	-0.01	-0.04	-0.06		14.2	20.3	-32.5	-0.38
w_O	0.34	0.68	0.35	-0.52	-0.01	-0.21	-0.64		-0.08	-65.2	79.2	-14.6
w_K	0.01	0.03	-0.01	-0.01	0.00	-0.01	-0.02		0.01	1.67	-0.64	-0.02
x_A	-0.18	0.22	0.08	-0.04	1.02	-0.08	-0.01		13.6	24.1	-37.4	0.41
x_I	-0.65	-1.07	-0.62	0.72	0.40	-1.37	3.55		5.73	123	-149	23.2
x_M	1.60	2.49	1.39	-2.88	-1.23	3.79	-4.08		-17.5	-329	413	-74.3
x_S	0.11	0.24	0.08	1.15	0.06	-0.46	-0.19		0.82	31.3	-48.1	17.0

Table 5. Comparative Static Results CES = 0.1

	v_G	v_P	v_C	v_L	v_R	v_O	v_K		p_A	p_I	p_M	p_S
w_G	-8.91	2.09	0.87	1.71	-0.17	0.47	3.88		-0.26	-5.49	5.27	1.38
w_P	1.31	-7.43	1.13	1.05	-0.08	1.20	2.80		-0.13	-11.1	12.2	-0.16
w_C	1.47	3.20	-8.60	3.76	0.45	0.12	-0.43		0.66	-3.57	-0.11	4.04
w_L	1.92	3.83	1.67	-2.71	-0.02	-1.14	-3.49		-0.02	3.23	-11.3	9.36
w_R	-1.60	2.48	0.91	-0.74	-0.05	-0.41	-0.56		14.2	20.3	-32.5	-0.38
w_O	3.44	6.84	3.54	-5.18	-0.06	-2.09	-6.37		-0.08	-65.2	79.2	-14.6
w_K	0.14	0.27	-0.07	-0.08	0.01	-0.06	-0.20		0.01	1.67	-0.64	-0.02
x_A	-0.18	0.22	0.08	-0.04	1.02	-0.08	-0.01		1.36	2.41	-3.74	0.04
x_I	-0.65	-1.07	-0.62	0.72	0.40	-1.37	3.55		0.57	12.3	-14.9	2.32
x_M	1.60	2.49	1.39	-2.88	-1.23	3.79	-4.08		-1.75	-32.9	41.3	-7.42
x_S	0.11	0.24	0.08	1.15	0.06	-0.46	-0.19		0.08	3.13	-4.81	1.70

The strongest negative $\hat{w}/\hat{p}$ effects occur for professional wages w_P and operator wages w_O with respect to industry prices p_I as well as for capital w_K and sales wages w_L with respect to manufacturing prices p_M. Elasticities of factor prices with respect to endowments $\hat{w}/\hat{v}$ scale inversely with CES substitution while the production frontier $\hat{x}/\hat{p}$ elasticities scale directly.

The typically low levels of substitution in the applied production literature suggest CES = 0.1 presents more reasonable model elasticities. Exogenous vectors of price changes can be introduced to simulate factor price adjustments. As an example, if the price of agricultural products p_A falls 10% and the price of services p_S rises 10%, the manager wage w_G would increase 16.4% while the professional wage w_P would fall 0.3%. Exogenous price changes may be due to taxes or tariffs on traded products as well as changes in international prices.

6. Conclusion

The present paper shows that comparative static price elasticities in general equilibrium production models are identical for any degree of CES substitution, from near zero to near perfect. These properties suggest applications of a wide range of general equilibrium models of production and trade can proceed without specification or estimation of factor substitution.

In even models with the same number of factors and products, the absence of endowment effects on factor prices is the factor price equalization property. Substitution plays no role either in the price elasticities of factor prices or the endowment elasticities of outputs. The output effects of price changes are diminished with a higher degree of CES substitution.

With more factors than products including specific factors models, a higher degree of CES substitution implies smaller differences in factor prices across countries with substitution increasing adjustment flexibility. Output effects of price changes also diminish with higher CES substitution. Price elasticities of factor prices are unaffected by the degree of CES substitution, as are elasticities of outputs with respect to factor endowments.

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