BASES OF TRANSLATES AND MULTIRESOLUTION ANALYSES [1]

Richard A. Zalik, Auburn University

Abstract. Using the theory of basis generators we study various properties of multivariate Riesz and orthonormal sequences of translates, with emphasis on those associated with multiresolution analyses and their connection with wavelets. In particular, we show that every multiresolution analysis of multiplicity n generated by a dilation matrix preserving the lattice $\mathbb{Z}^d$ has an orthonormal wavelet system associated with it, and give a closed form representation in Fourier space for such wavelet systems. We illustrate these results by applying them to the case of univariate wavelets associated with multiresolution analyses with binary dilations.

Keywords: Schauder, Riesz and orthogonal bases of translates; basis generators; dilation matrices; wavelet systems; multiresolution analyses of multiplicity n.

[1]Most of the results that appear in this paper were presented at a research seminar at Universidad Nacional de Mar del Plata, Argentina. The author wishes to thank the institution for its generous support and faculty members Alfredo L. González and M. del Carmen Moure for their helpful comments.

Author's address: Department of Mathematics and Statistics, Auburn University, AL 36849-5310. E-mail: zalik@auburn.edu

This file differs from the printed version in that a number of misprints have been corrected.

1. Introduction

The main results of this paper (Theorems 9 and 12) give explicit closed form representations in Fourier space for all multivariate orthonormal and semiorthogonal Riesz wavelet systems associated with a given multiresolution analysis. Such formulas were only known for the univariate case (see Theorems A and B). These representation theorems did not grow on fallow ground; rather, they developed in a field tilled by many other people. We therefore felt it would be appropriate to review some related results in order to provide context, and also because they are widely scattered in the literature. In addition to the main results mentioned above, we also present some properties of bases of translates that are of independent interest.

In what follows, $\mathbb{Z}$ will denote the set of integers and $\mathbb{Z}_+$ the set of all strictly positive integers. The underlying space will be $L^2(\mathbb{R}^d)$, where $d \geq 1$ is an integer and $\mathbb{R}$ is the set of real numbers; $\mathbb{C}$ will denote the set of complex numbers, and $\mathbf{I}$ will stand for the identity matrix. Boldface lowercase letters will denote elements of $\mathbb{R}^d$, which will be represented as column vectors; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g \rangle$, their bracket product by $[f, g]$, and the norm of f by $||f||$; thus,

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(\mathbf{t}) \overline{g(\mathbf{t})} \, d\mathbf{t},$$

$$[f, g](\mathbf{t}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} f(\mathbf{t} + \mathbf{k}) \overline{g(\mathbf{t} + \mathbf{k})},$$

and

$$||f|| := \sqrt{\langle f, f \rangle}.$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi \mathbf{x} \cdot \mathbf{t}} f(\mathbf{t}) \, d\mathbf{t}.$$

Let $\mathbf{A} \in \mathbb{C}^{d \times d}$ and $a := \det \mathbf{A}$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D_j^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D_j^{\mathbf{A}} f(\mathbf{t}) := |a|^{j/2} f(\mathbf{A}^j \mathbf{t})$$

and

$$T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{k}).$$

In particular, if $d = 1$

$$D_j f(t) := 2^{j/2} f(2^j t).$$

Let $\mathbb{T} := [0, 1]$, and let $\mathbb{T}^d$ denote the d–dimensional torus. A function f will be called $\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$. A set $S \subset L^2(\mathbb{R}^d)$ is called *shift–invariant* if $f \in S$ implies that $T_{\mathbf{k}} f \in S$ for every $\mathbf{k} \in \mathbb{Z}^d$.

Let $\mathbf{u} \subset L^2(\mathbb{R}^d)$; then

$$T(\mathbf{u}) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}$$

and

$$S(\mathbf{u}) := \overline{\operatorname{span}} \, T(\mathbf{u}),$$

where the closure is in $L^2(\mathbb{R}^d)$. If $\mathbf{u} = \{u_1, \cdots, u_m\}$ then $S(\mathbf{u})$ is called a *finitely generated shift–invariant space* or FSI and the functions u_ℓ are called the *generators* of $S(\mathbf{u})$. In this case we will also use the symbols $T(u_1, \cdots, u_n)$ and $S(u_1, \cdots, u_n)$ to denote $S(\mathbf{u})$ and $T(\mathbf{u})$ respectively. If $\mathbf{u}$ contains a single element, then $S(\mathbf{u})$ is called a *principal shift–invariant space* or PSI.

Theorems, lemmas and propositions that are either known or are trivial corollaries of known results, are labeled using the letters of the alphabet.

Recall that a wavelet $\psi \in L^2(\mathbb{R})$ is called *semiorthogonal* if $j \neq q$ implies that $\langle D_j T_k \psi_\ell, D_q T_n \psi_r \rangle = 0$ for every $k, n \in \mathbb{Z}$.

Our starting point and motivation are the following characterizations in Fourier space of MRA orthonormal wavelets and MRA semiorthogonal Riesz wavelets in $L^2(\mathbb{R})$ which, with the definition of Fourier transform we

have adopted, may be stated as follows.

Theorem A. *Let φ be a scaling function for M with associated low pass filter p. The following propositions are equivalent:*
(a) ψ is an MRA orthonormal wavelet associated with M.
(b) There is a measurable unimodular $\mathbb{Z}$–periodic function $\mu(x)$ such that

$$\widehat{\psi}(2x) = e^{i2\pi x}\mu(2x)\overline{p(x+1/2)}\widehat{\varphi}(x) \qquad a.e. \tag{1}$$

Theorem B. *A function ψ is a semiorthogonal Riesz wavelet with bounds $0 < A \le B$ associated with an MRA M if and only if there is a measurable $\mathbb{Z}$–periodic function $\mu(x)$ and an orthonormal wavelet h associated with M such that*

$$A \le |\mu(x)|^2 \le B \quad a.e.$$

and

$$\widehat{\psi}(x) = \mu(x)\widehat{h}(x) \qquad a.e.$$

Theorem A is well known (see e.g. Hernández and Weiss [17], Wojtaszczyk [30]). Theorem B was proved by the author ([32, Corollary 2]). Theorems 9 and 12 below generalize these theorems to orthogonal and semiorthogonal Riesz wavelet systems in $L^2(\mathbb{R}^d)$.

2. Bessel sequences, frames, and Riesz bases of translates

Let $\mathbb{H}$ be a (separable) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$. A sequence $F = \{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is called a *frame* if there are constants $0 < A \le B$ such that for every $f \in \mathbb{H}$

$$A\|f\|^2 \le \sum_{\mathbf{k}\in\mathbb{Z}^d} |\langle f, f_k\rangle|^2 \le B\|f\|^2.$$

The constants A and B are called (lower and upper) *bounds* of the frame. If only the right-hand inequality in the preceding displayed formula is satisfied for all $f \in \mathbb{H}$, then F is called a *Bessel sequence* with bound B. It is called a *Riesz sequence* if there are constants A and B, $A > 0$, such that for every sequence $\{c_k, k \in \mathbb{Z}\} \subset \ell^2$

$$A\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_k|^2 \le \left\|\sum_{\mathbf{k}\in\mathbb{Z}^d} c_k f_k\right\|^2 \le B\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_k|^2.$$

Finally, F is called a *Riesz basis* of $\mathbb{H}$ if it is a Riesz sequence and its linear span is dense in $\mathbb{H}$. The constants A and B are called (lower and upper) *bounds* of the Riesz basis. Every Riesz basis is a frame, every orthonormal basis is a Riesz basis with bounds $A = B = 1$, and Riesz bounds and frame bounds coincide. A sequence $\{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is a Riesz basis if and only if it is a frame having the additional property that upon the removal of any element from the sequence, it ceases to be a frame [2, 13, 31].

Given a sequence of functions $\mathbf{u} := \{u_1, \cdots, u_m\}$ in $L^2(\mathbb{R}^d)$, by $G[u_1, \cdots, u_m](\mathbf{x})$, $G_{\mathbf{u}}(\mathbf{x})$ or $G(\mathbf{x})$ we will denote its *Gramian matrix*, viz.

$$G(\mathbf{x}) := \Big([\widehat{u_\ell}, \widehat{u_j}](\mathbf{x})\Big)_{\ell,j=1}^m.$$

The range function $J_S(\mathbf{x})$ associated with S is defined by

$$J_S(\mathbf{x}) := \overline{\operatorname{span}}\,\{(f(\mathbf{x}+\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^d}; f \in S(\mathbf{u})\}$$

and the *spectrum* $\sigma(\mathbf{u})$ of $\mathbf{u}$ by

$$\sigma(\mathbf{u}) := \{\mathbf{x} \in \mathbb{T}^d; \dim J_S(\mathbf{x}) \ne 0\}.$$

Let $\Lambda \subset \mathbb{Z}$ and $\mathbf{u} = \{u_k; k \in \Lambda\} \subset S \subset L^2(\mathbb{R}^d)$. If S is a shift–invariant space then $\mathbf{u}$ is called a *basis generator* of S, and we say that $\mathbf{u}$ *provides a basis* for S, if for every $f \in S$ there are $\mathbb{Z}$–periodic functions p_k, uniquely determined by f (up to a set of measure 0), such that

$$\widehat{f} = \sum_{k\in\Lambda} p_k\widehat{u_k}.$$

If $\mathbf{u}$ is a finite set, the uniqueness of the functions p_k is equivalent to $G_{\mathbf{u}}(\mathbf{x})$ being nonsingular for almost every $\mathbf{x} \in \mathbb{T}^d$ [4, 5].

The theory of basis generators has been extensively developed by De Boor, DeVore, Ron and Shen in [4, 5, 25] (where sometimes they are just called "bases"). Here we shall apply some of these results to the study of *Schauder basis generators*, *Riesz basis generators* and *orthonormal basis generators*, i.e. sets $\mathbf{u}$ such that $T(\mathbf{u})$ is either a Schauder basis, a Riesz basis, or an orthonormal basis of $S(\mathbf{u})$. Note that a Riesz basis generator is a basis generator.

Let $\Lambda(\mathbf{x})$, $\lambda^+(\mathbf{x})$ and $\lambda(\mathbf{x})$ respectively denote the largest eigenvalue of $G(\mathbf{x})$, the smallest nonzero eigenvalue of $G(\mathbf{x})$, and the smallest eigenvalue of $G(\mathbf{x})$. It follows from the properties of Hermitian positive semidefinite matrices that $\Lambda(\mathbf{x}) = ||G(\mathbf{x})||$, and that if $G(\mathbf{x})$ is nonsingular, then

$$(\lambda(\mathbf{x}))^{-1} = ||G(\mathbf{x})^{-1}||,$$

where "$||\cdot||$" is the matrix norm induced by the euclidean vector norm. Thus, if ρ denotes the spectral radius, we have $\Lambda(\mathbf{x}) = \rho(G(\mathbf{x}))$ and, if $G(\mathbf{x})$ is nonsingular,

$$(\lambda(\mathbf{x}))^{-1} = \rho(G^{-1}(\mathbf{x})).$$

In [25, Theorem 2.3.6] Ron and Shen provided characterizations of Bessel sequences, frames, and Riesz bases of translates in FSI spaces (see also Bownik [6]). These conditions, as well as the corresponding Bessel, frame and Riesz bounds, are expressed in terms of these induced matrix norms or in terms of $\Lambda(\mathbf{x})$, $\lambda^+(\mathbf{x})$ and $\lambda(\mathbf{x})$. Although the bounds given are optimal ([5, Lemma 3.27]), it is relatively difficult to verify that the conditions of these theorems are satisfied. It is therefore useful to have alternative conditions to characterize these sequences, that are easier to verify. For Riesz sequences this was done by Lemarié–Rieusset [20, Lemme 2].

Proposition C. *Assume that* $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)\}$. *Then*

(a) $T(\mathbf{u})$ *is a Bessel sequence if and only if for every* $\ell, j = 1, \cdots, m$ *the coefficients* $[\widehat{u_\ell}, \widehat{u_j}]$ *are essentially bounded.*

(b) $T(\mathbf{u})$ *is a Riesz sequence if and only if for every* $\ell, j = 1, \cdots, m$ *the coefficients* $[\widehat{u_\ell}, \widehat{u_j}](\mathbf{x})$ *are essentially bounded, and there is a constant* $C > 0$ *such that* $\det[G(\mathbf{x})] \geq C > 0$ *for almost every* $\mathbf{x} \in \mathbb{T}^d$.

Part (b) of Proposition C is [20, Lemme 2]; the proof of part (a) is implicit in the proof of part (b).

It is easy to see that the boundedness of the coefficients $[\widehat{u_\ell}, \widehat{u_j}]$ in Proposition C can be replaced by the assertion that the trace $\mathrm{tr}[G_{\mathbf{u}}(\mathbf{x})]$ be bounded. Indeed, assume there is a constant K such that $\mathrm{tr}[G_{\mathbf{u}}(\mathbf{x})] \leq K$ for almost every $\mathbf{x} \in \mathbb{T}^d$. Note that $\mathrm{tr}[G_{\mathbf{u}}(\mathbf{x})]$ equals the sum of all eigenvalues of $G_{\mathbf{u}}(\mathbf{x})$. Moreover, since $G_{\mathbf{u}}(\mathbf{x})$ is positive semidefinite all these eigenvalues are nonnegative. We therefore conclude that $\Lambda(\mathbf{x}) \leq \mathrm{tr}[G_{\mathbf{u}}(\mathbf{x})]$, and the assertion follows from [25, Theorem 2.3.6]. Conversely, assume $T(\mathbf{u})$ is a Bessel sequence with bound B. Then $\mathrm{tr}[G_{\mathbf{u}}(\mathbf{x})] \leq m\Lambda(\mathbf{x}) \leq mB$.

Let

$$p_\ell(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} a_{\ell,\mathbf{k}} e^{-i2\pi \mathbf{k}\cdot\mathbf{x}}.$$

The proofs of both [20, Lemme 2] and [25, Theorem 2.3.6] make use of the identity

$$||\sum_{\ell=1}^{m} \sum_{\mathbf{k} \in \mathbb{Z}^d} a_{\ell,\mathbf{k}} T_{\mathbf{k}} u_\ell||^2_{L^2(\mathbb{R}^d)} = ||(p_1, \cdots, p_m)\, G_{\mathbf{u}}\, (p_1, \cdots, p_m)^T||^2_{L^2(\mathbb{T}^d)};$$

see also Bownik [6, p.294].

Since $G_{\mathbf{u}}(\mathbf{x})$ is $\mathbb{Z}^d$–periodic and $\mathbb{T}^d$ is a compact set, we have the following analog of [5, Theorem 3.19(c)]

Corollary 1. *Assume that* $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$ *is such that the functions* $[\widehat{u_\ell}, \widehat{u_j}](\mathbf{x})$, $\ell, j = 1, \cdots, m$ *are continuous on* $\mathbb{T}^d$. *Then*

(a) $T(\mathbf{u})$ *is a Bessel sequence.*

(b) $T(\mathbf{u})$ *is a Riesz sequence if and only if* $G_{\mathbf{u}}(\mathbf{x})$ *is nonsingular for every* $\mathbf{x} \in \mathbb{T}^d$.

Given a shift–invariant space S, let *len* S denote the smallest number k such that S has a set of k generators. We will use this definition in the proof of

Theorem 1. *Let* $\mathbf{u} := \{u_1, \cdots, u_n\}$ *and* $\mathbf{v} := \{v_1, \cdots, v_m\}$.

(a) *If* $T(\mathbf{u})$ *and* $T(\mathbf{v})$ *are Riesz bases of the same shift–invariant space* $S \subset L^2(\mathbb{R}^d)$, *then* $n = m$.

(b) *If* $T(\mathbf{u})$ *and* $T(\mathbf{v})$ *are Riesz sequences such that* $n = m$ *and* $S(\mathbf{u}) \subset S(\mathbf{v})$, *then* $T(\mathbf{u})$ *is a Riesz basis of* $S(\mathbf{v})$.

(c) *Let* $T(\mathbf{u})$ *and* $T(\mathbf{v})$ *be Riesz sequences in* $L^2(\mathbb{R}^d)$, *and assume that* $S(\mathbf{u})$ *is a proper subset of* $S(\mathbf{v})$. *Then* $n < m$ *and there are functions* $w_1, \cdots, w_{m-n}$ *such that*

$$T(w_1, \cdots, w_{m-n})$$

is an orthonormal basis of the orthogonal complement $S(\mathbf{u})^\perp$ *of* $S(\mathbf{u})$ *in* $S(\mathbf{v})$, *and*

$$T(u_1, \cdots, u_n, w_1, \cdots, w_{m-n})$$

is a Riesz basis of $S(v_1, \cdots, v_m)$.

Proof. (a) is a consequence of [4, Theorem 2.26(ii)]. If $T(\mathbf{u})$ is a Riesz basis then $\mathbf{u}$ is a basis generator, and therefore $G_{\mathbf{u}}(\mathbf{x})$ is nonsingular a.e.; thus (b) is a consequence of [4, Theorem 2.26(iii)]. To prove (c) we proceed as follows: assume that $n \geq m$; then $T(u_1, \cdots, u_m)$ is a Riesz sequence and $S(u_1, \cdots, u_m)$ is a proper subset of $S(\mathbf{v})$, in contradiction of (b); thus $n < m$. The hypotheses also imply that $\sigma(\mathbf{u}) = \sigma(\mathbf{v}) = \mathbb{T}^d$. From [4, Theorem 2.26(ii)] we infer that *len* $S(\mathbf{u}) = n$ and *len* $S(\mathbf{v}) = m$; thus [5, Theorem 3.13] implies that $\sigma(S(\mathbf{u})^\perp) = \mathbb{T}^d$ and that *len* $S(\mathbf{u})^\perp = m - n$. From [5, Corollary 3.4] we deduce that $S(\mathbf{u})^\perp$ is a closed shift–invariant space. Thus [5, Lemma 3.6] implies that $S(\mathbf{u})^\perp$ contains a principal shift invariant space S_1 such that $\sigma(S_1) = \mathbb{T}^d$. If $\widetilde{S_1}$ denotes the orthogonal complement of S_1 in $S(\mathbf{u})^\perp$, [5, Theorem 3.13] implies that $\sigma(\widetilde{S_1}) = \mathbb{T}^d$ and *len* $\widetilde{S_1} = m - n - 1$. Repeating this procedure we conclude that $S(\mathbf{u})^\perp$ is the orthogonal sum of $m - n$ principal space invariant spaces S_k such that $\sigma(S_k) = \mathbb{T}^d$, $k = 1, \cdots, m - n$. Therefore [5, Theorem 2.10] implies that there are functions w_k such that $T(w_k)$ is an orthonormal basis of S_k. By orthogonality we conclude that $T(w_1, \cdots, w_r)$ is an orthonormal basis of $S(\mathbf{u})^\perp$. Moreover,

$$[\widehat{u_\ell}, \widehat{w_k}] = 0; \quad \ell = 1, \cdots, n, \ k = 1, \cdots, m - n$$

and

$$[\widehat{w_\ell}, \widehat{w_k}] = \delta_{\ell,k}; \quad \ell, k = 1, \cdots, m - n.$$

If $y_\ell := u_\ell$ for $\ell = 1, \cdots, n$, $y_\ell := w_{\ell-n}$ for $\ell = n + 1, \cdots, r$, $\mathbf{y} := \{y_1, \cdots, y_m\}$, and for each $\mathbf{x} \in \mathbb{T}^d$, $E_{\mathbf{u}}(\mathbf{x})$ and $E_{\mathbf{y}}(\mathbf{x})$ denote the sets of eigenvalues of $G_{\mathbf{u}}(\mathbf{x})$ and $G_{\mathbf{y}}(\mathbf{x})$ respectively, then $E_{\mathbf{y}}(\mathbf{x}) = E_{\mathbf{u}}(\mathbf{x}) \cup \{1\}$. It follows from [25, Theorem 2.3.6] that $T(\mathbf{y})$ is a Riesz sequence in $S(\mathbf{v})$, and (b) implies that $T(\mathbf{y})$ is a Riesz basis of $S(\mathbf{v})$. □

Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$; that is, $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$ and all its eigenvalues have modulus greater than 1. These conditions imply that $\mathbf{A} \in \mathbb{Z}^{d\times d}$, and that if $a := \det \mathbf{A}$ then $|a|$ is an integer larger than 1. A *multiresolution analysis* (MRA) of multiplicity n in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}$, $f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{A}\mathbf{t}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) There are functions $\mathbf{u} := \{u_1, \cdots, u_n\}$ such that $T(\mathbf{u})$ is an orthonormal basis of V_0.

The condition that $T(\mathbf{u})$ be an orthonormal basis may be replaced by the condition that $T(\mathbf{u})$ be a Riesz basis (see Proposition G below).

It follows from the definition of multiresolution analysis that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j} \in L^2(\mathbb{T}^d)$ such that the functions u_ℓ satisfy the *scaling identity*

$$\widehat{u_\ell}(\mathbf{A}^*\mathbf{x}) = \sum_{j=1}^{n} p_{\ell,j}(\mathbf{x})\widehat{u_j}(\mathbf{x}), \quad j,\ell = 1,\cdots,n \quad a.e.,$$

where $\mathbf{A}^*$ is the conjugate transpose of $\mathbf{A}$. The functions u_ℓ are called *scaling functions* for the multiresolution analysis, and the functions $p_{\ell,j}$ are called the *low pass filters* associated with $\mathbf{u}$.

In [3, Corollary 2.3] Bownik and Garrigós showed that the multiplicity of a multiresolution analysis is a constant that depends only on V_0 and not on the choice of scaling functions. This also follows from Theorem 1.

From the definition of the dilation operator at the beginning of Section 1 we see that

$$D_1^{\mathbf{A}} f(\mathbf{t}) = |a|^{1/2} f(\mathbf{A}\mathbf{t}),$$

where $a := \det \mathbf{A}$. If $\{\mathbf{A}_1,\cdots,\mathbf{A}_m\} \subset \mathbb{C}^{d\times d}$, let

$$T(\mathbf{A}_1,\cdots,\mathbf{A}_m;u_1,\cdots,u_m) := \{D_1^{\mathbf{A}_\ell} T_{\mathbf{k}} u_\ell; \ell = 1,\cdots m, \mathbf{k} \in \mathbb{Z}^d\}, \tag{2}$$

and

$$S(\mathbf{A}_1,\cdots,\mathbf{A}_m;u_1,\cdots,u_m) := \overline{\operatorname{span}}\, T(\mathbf{A}_1,\cdots,\mathbf{A}_m;u_1,\cdots,u_m). \tag{3}$$

If

$$\mathbf{A}_\ell = \mathbf{A}, \quad \ell = 1,\cdots,m, \quad \text{and} \quad \mathbf{u} = \{u_1,\cdots,u_m\}$$

we will write $T(\mathbf{A};\mathbf{u})$ or

$$T(\mathbf{A};u_1,\cdots,u_m)$$

instead of (2), and $S(\mathbf{A};\mathbf{u})$ or

$$S(\mathbf{A};u_1,\cdots,u_m)$$

instead of (3). Note that $T(\mathbf{u}) = T(\mathbf{I};\mathbf{u})$ and $S(\mathbf{u}) = S(\mathbf{I};\mathbf{u})$.

Theorem 2. *Let*

$$\{\mathbf{A}_1,\cdots,\mathbf{A}_m\} \subset \mathbb{Z}^{d\times d} \quad \textit{and} \quad \mathbf{u} = \{u_1,\cdots,u_m\} \subset L^2(\mathbb{R}^d).$$

Assume that for some ℓ, $1 \le \ell \le m$, $\mathbf{A}_\ell\mathbb{Z}^d \neq \mathbb{Z}^d$. If there are functions $\varphi_1,\cdots,\varphi_m$ such that $T(\varphi_1,\cdots,\varphi_m)$ is a Riesz sequence and

$$T(\mathbf{A}_1,\cdots,\mathbf{A}_m;u_1,\cdots,u_m) \tag{4}$$

is a Riesz sequence in $S := S(\varphi_1,\cdots,\varphi_m)$, then (4) cannot be a Riesz basis of S.

Proof. Assume that (4) is a Riesz basis of $S(\varphi_1,\cdots,\varphi_m)$ and $v_\ell := D_1^{\mathbf{A}_\ell} u_\ell$. Then $T(v_1,\cdots v_m)$ is a proper subsequence of (4) and also a Riesz sequence. From Theorem 1 we deduce that $T(v_1,\cdots v_m)$ is a basis of $S(\varphi_1,\cdots,\varphi_m)$, and we have a contradiction. □

Corollary 2. *Let $\{j_1,\cdots,j_m\}$ be a set of nonnegative integers, not all of them equal to 0, let $T(\mathbf{u})$ be a Riesz sequence, and assume that $\mathbf{A} \in \mathbb{R}^{d\times d}$ is such that*

$$\mathbf{A}\mathbb{Z}^d \neq \mathbb{Z}^d.$$

Then there is no set of functions $\{\varphi_1,\cdots,\varphi_m\}$ such that

$$T(\mathbf{A}^{j_1},\cdots,\mathbf{A}^{j_m};\varphi_1,\cdots\varphi_m)$$

is a Riesz basis of $S(u_1,\cdots,u_m)$.

If $\mathbf{A}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$ then, in the notation used by Madych in [21], a coset of $\mathbf{A}\mathbb{Z}^d$ is a set of the form

$$\mathbf{j} + \mathbf{A}\mathbb{Z}^d = \{\mathbf{j} + \mathbf{A}\mathbf{r}; \mathbf{r} \in \mathbb{Z}^d\},$$

where $\mathbf{j} \in \mathbb{Z}^d$. An element of a coset is called a *representative* of the coset. Any pair of cosets are either identical or disjoint, and the union of all disjoint cosets equals $\mathbb{Z}^d$. There are exactly $|a|$ disjoint cosets. (cf. Wojtaszczyk [30]). The collection of all disjoint cosets is denoted by $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$. A set $\mathbf{J} \subset \mathbb{Z}^d$ is said to be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$ if it contains exactly $|a|$ elements and

$$\bigcup_{\mathbf{j}\in\mathbf{J}} (\mathbf{j} + \mathbf{A}\mathbb{Z}^d) = \mathbb{Z}^d.$$

For example, if $\mathbf{A} = 2\mathbf{I} \in \mathbb{Z}^{d\times d}$ then $|a| = 2^d$, and a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$ is the set of all vectors in $\mathbb{Z}^d$ whose coordinates equal either 0 or 1.

Starting with a shift–invariant space $S(\mathbf{u})$ having a set of n orthonormal basis generators and a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$, we now focus on how to construct an orthonormal basis generator of $S(\mathbf{A}; \mathbf{u})$. If $\mathbf{J}$ is a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$ it is easy to see that $\{D_1^{\mathbf{A}} T_j u_\ell; 1 \le \ell \le n, j \in \mathbf{J}\}$ is such a basis generator. However, the indices in this sequence are elements of the cartesian product $[1, n] \times \mathbf{J}$. We find it more convenient to use a slightly different formulation in which the indices are linearly ordered.

Theorem 3. *Let $\mathbf{u} = \{u_1, \cdots, u_n\} \subset L^2(\mathbb{R}^d)$ and assume that $T(\mathbf{u})$ is an orthonormal sequence. Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, $a := \det \mathbf{A}$, and let $\mathbf{J}$ be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$. For $\mathbf{x} > 0$ define $I(x) := [1, x] \cap \mathbb{Z}$, and let*

$$p = (p_1, p_2) : I(|a|n) \longrightarrow I(n) \times \mathbf{J}$$

be a bijection. If

$$v_\ell(t) := |a|^{1/2} u_{p_1(\ell)}(\mathbf{A}t + p_2(\ell))$$

and $\mathbf{v} := \{v_1, \cdots, v_{|a|n}\}$, then $T(\mathbf{v})$ is an orthonormal basis of $S(\mathbf{A}; \mathbf{u})$, and every Riesz basis generator of $S(\mathbf{A}; \mathbf{u})$ has exactly $|a|n$ functions.

Proof. Let $1 \le \ell \le |a|n$, $\mathbf{r} \in \mathbb{Z}^d$, $s := p_1(\ell)$ and $\mathbf{j} := p_2(\ell)$. Define $\mathbf{k} := \mathbf{A}\mathbf{r} + \mathbf{j}$. From the hypotheses we know that $\mathbf{k} \in \mathbb{Z}^d$. From the discussion that preceded the statement of this theorem and the nonsingularity of $\mathbf{A}$ we see that $\mathbf{k}$ uniquely determines $\mathbf{j}$ and $\mathbf{r}$. Since $T(\mathbf{u})$ is an orthonormal sequence, a trivial change of variables shows that also $T(A; \mathbf{u})$ is an orthonormal sequence. But $T_{\mathbf{r}} v_\ell = D_1^{\mathbf{A}} T_{\mathbf{k}} u_s$, and we deduce that $T(\mathbf{v})$ is an orthonormal sequence in $S(\mathbf{A}; \mathbf{u})$. Moreover, if $1 \le s \le n$ and $\mathbf{k} \in \mathbb{Z}^d$ there are (unique) $\mathbf{r} \in \mathbb{Z}^d$ and $1 \le \ell \le |a|n$ such that $D_1^{\mathbf{A}} T_{\mathbf{k}} u_s = T_{\mathbf{r}} v_\ell$; thus $T(\mathbf{v})$ is an orthonormal basis of $S(\mathbf{A}; \mathbf{u})$.

Since $\mathbf{v}$ has exactly $|a|n$ functions, Theorem 1 implies that every other Riesz basis generator of $S(\mathbf{A}; \mathbf{u})$ must have exactly $|a|n$ functions. □

If $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity n and $\mathbf{u}$ is an orthonormal basis generator of V_0, then $V_j = S(\mathbf{A}^j; \mathbf{u})$. Therefore Theorem 1(c) and Theorem 3 yield

Corollary 3. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$, and let W_0 denote the orthogonal complement of V_0 in V_1. Then W_0 has a Riesz basis generator, and every Riesz basis generator of W_0 has exactly $(|a| - 1)n$ functions.*

Theorem 1 raises the following question: if U has a Riesz basis of translates generated by m functions, V has a Riesz basis of translates generated by n functions and U is a proper subset of V, how are m and n related? We study this question within the framework of a multiresolution analysis. Applying Theorem 3 we obtain

Theorem 4. *Let $\{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$, and let $a := \det \mathbf{A}$. Then*

(a) *If $r > 0$ then V_r has a finite Riesz basis generator.*
(b) *For any $r \in \mathbb{Z}$, any finite Riesz basis generator of V_r must have exactly $n|a|^r$ functions.*

Proof. For $r > 0$ the existence of a Riesz basis generator of $n|a|^r$ functions follows by repeated application of Theorem 3. Assume therefore that $r < 0$ and that V_r has a finite Riesz basis generator of k functions. By repeated application of Theorem 3 we deduce that V_0 has a Riesz basis generator of $k|a|^{-r}$ functions. But Theorem 1 implies that $k|a|^{-r} = n$, and the assertion follows. □

It follows from Theorem 4 that V_r cannot have a Riesz basis generator unless $n|a|^r$ is an integer. In particular, we have

Corollary 4. *Let $M = \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity 1 generated by $\mathbf{A} := 2\mathbf{I}$. If $r < 0$, then V_r cannot have a Riesz basis generator.*

Theorem 4 also yields

Corollary 5. *Let $\{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$, let $r > 0$, and $U_j := V_{j+r}$. Then $\{U_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity $n|a|^r$.*

Corollary 6. *If $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity n in $L^2(\mathbb{R}^d)$ then V_j is a proper subset of V_{j+1} for every $j \in \mathbb{Z}$.*

Proof. If $j \geq 0$ the assertion follows from Theorem 4. Assume that there is an integer $j < 0$ such that $V_j = V_{j+1}$. We can no longer apply Theorem 4, since V_j may not have a Riesz basis generator. We bypass this difficulty by noting that the definition of multiresolution analysis implies that $V_j = V_{j+k}$ for every $k > 0$, and in particular that $V_0 = V_1$, which is impossible. □

3. Linear Transformations

At this point, it is convenient to state two auxiliary lemmas and a theorem that will be used in the subsequent discussion.

Lemma D. *$T(u_1, \cdots, u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ if and only if $G_{\mathbf{u}}(\mathbf{x}) = \mathbf{I}$ a.e.*

Proof. From e.g. [4, Lemma 2.7] we see that $S(v) \perp S(w)$ if and only if $[\widehat{v}, \widehat{w}](\mathbf{x}) = 0$ a.e. Moreover, $T(v)$ is an orthonormal sequence if and only if $[\widehat{v}, \widehat{v}](\mathbf{x}) = 1$ a.e. □

Lemma E. *Let $T(u)$ and $T(v_1, \cdots, v_m)$ be orthonormal sequences in $L^2(\mathbb{R}^d)$, and assume there are $\mathbb{Z}^d$–periodic functions p_ℓ such that*

$$\widehat{u}(\mathbf{x}) = \sum_{\ell=1}^{m} p_\ell(\mathbf{x})\widehat{v_\ell}(\mathbf{x}) \quad a.e.$$

Then

$$\sum_{\ell=1}^{m} |p_\ell(\mathbf{x})|^2 = 1 \quad a.e.$$

Proof. Since both sequences are orthonormal, we have

$$1 = [\widehat{u}, \widehat{u}](\mathbf{x}) = [\sum_{\ell=1}^{m} p_\ell \widehat{v_\ell}, \sum_{\ell=1}^{m} p_\ell \widehat{v_\ell}](\mathbf{x}) = \sum_{\ell=1}^{m}\sum_{j=1}^{m} p_\ell(\mathbf{x})\overline{p_j(\mathbf{x})}[\widehat{v_\ell}, \widehat{v_j}](\mathbf{x}) = \sum_{\ell=1}^{m} |p_\ell(\mathbf{x})|^2(\mathbf{x}) \quad a.e.$$

□

A proof of Lemma D may be found in e.g. Soardi [27]. We have included proofs of Lemmas D and E for the sake of completeness.

The following is a straightforward corollary of [4, Theorem 1.7].

Theorem F. *Let* $\mathbf{h} = \{h_1, \cdots, h_m\}$ *and assume that* $S(\mathbf{u}) = S(u_1, \cdots, u_n) \subset S(\mathbf{h})$. *Then there are* $\mathbb{Z}^d$*–periodic functions* $q_{\ell,j}(\mathbf{x})$ *such that*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{m} q_{\ell,j}(\mathbf{x})\widehat{h_j}(\mathbf{x}) \quad a.e.; \quad \ell = 1, \cdots, n.$$

The matrix

$$\mathbf{Q}(\mathbf{x}) := \Big(q_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{n,m}$$

will be called a *transition matrix* from the sequence $T(\mathbf{h})$ to the sequence $T(\mathbf{u})$. If $\mathbf{h}$ is a basis generator of $S(\mathbf{h})$, then $Q(\mathbf{x})$ is unique (up to a set of measure 0).

In what follows we assume that $T(\mathbf{h}) = T(h_1, \cdots, h_m) \subset L^2(\mathbb{R}^d)$ is a Schauder, orthonormal or Riesz basis of $S(\mathbf{u})$, and we study the effect of a linear transformation on such a set of functions.

A proof of the identity

$$G_{\mathbf{u}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})G_{\mathbf{h}}(\mathbf{x})\mathbf{Q}^*(\mathbf{x}) \tag{5}$$

can be found in e.g. [14]. As remarked in [4, Theorem 2.26], it is clear from (5) that if $T(\mathbf{h})$ is a Riesz sequence, then $T(\mathbf{u})$ is a Riesz basis of $S(\mathbf{h})$ if $n = m$ and both $||\mathbf{Q}(\mathbf{x})||$ and $||\mathbf{Q}^{-1}(\mathbf{x})||$ are in $L_\infty(\mathbb{T}^d)$. We now digress to mention a related result, of independent interest.

Recall that if $\mathbf{u}$ is a finite set of functions such that $T(\mathbf{u})$ is a Riesz basis, then $G_{\mathbf{u}}$ is positive definite for almost every $\mathbf{x} \in \mathbb{T}^d$. Thus the square root of $G_{\mathbf{u}}$ (i.e. the unique positive definite matrix $H_{\mathbf{u}}$ such that $H_{\mathbf{u}}^2 = G_{\mathbf{u}}$) exists for almost every $\mathbf{x} \in \mathbb{T}^d$.

If $T(u)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$ it is well known that if $\widehat{h} := [\widehat{u}, \widehat{u}]^{-1/2}\widehat{u}$, then $T(h)$ is an orthonormal basis of $S(u)$. A similar assertion is also true in the case of several generators: see Meyer [22, p.26, Lemme 3] or [23, p.25, Lemma 3], where a proof in the time domain is given. There is also a construction in Fourier space that naturally extends the single generator formulation.

Proposition G. *Let* $\mathbf{h} = \{h_1, \cdots, h_m\}$ *and assume that* $\mathbf{u} := \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$ *is such that* $T(\mathbf{u})$ *is a Riesz sequence. Let*

$$\mathbf{Q}(\mathbf{x}) := [G_{\mathbf{u}}(\mathbf{x})]^{-1/2}$$

and

$$(\widehat{h_1}(\mathbf{x}), \cdots, \widehat{h_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})(\widehat{u_1}(\mathbf{x}), \cdots, \widehat{u_m}(\mathbf{x}))^T.$$

Then $T(\mathbf{h})$ *is an orthonormal basis of* $S(\mathbf{u})$.

To prove this proposition, note that the hypothesis implies that $G_{\mathbf{u}}(\mathbf{x})$ is nonsingular for almost every $\mathbf{x} \in \mathbb{T}^d$. Moreover, since $G_{\mathbf{u}}(\mathbf{x})$ is Hermitian so is $\mathbf{Q}(\mathbf{x})$. Also, note that $\mathbf{Q}(\mathbf{x})$ is $\mathbb{Z}^d$–periodic. Applying (5) (with the roles of $\mathbf{h}$ and $\mathbf{u}$ reversed) we deduce that $G_{\mathbf{h}}(\mathbf{x}) = \mathbf{I}$ for almost every $\mathbf{x} \in \mathbb{T}^d$; thus Lemma D implies that $T(\mathbf{h})$ is an orthonormal sequence in $S(\mathbf{u})$. Applying Theorem 1 we conclude that, moreover, $T(\mathbf{h})$ is a basis of $S(\mathbf{u})$.

Proposition G and the preceding proof may be found in Ashino and Kametani [1], where it is attributed to an anonymous referee.

In summation, we know that if $\mathbf{u}$ is a Riesz basis of $S(\mathbf{u})$ then the same space has an orthonormal basis. It follows that condition (iv) in the definition of a multiresolution analysis may be replaced by the assumption that V_0 has a Riesz basis.

We now return to [4, Theorem 2.26]. As remarked in the discussion leading to Proposition C, the direct computation of induced matrix norms (such as, in our case, $||\mathbf{Q}(\mathbf{x})||$ and $||\mathbf{Q}^{-1}(\mathbf{x})||$) may be relatively difficult. We therefore seek alternative criteria that will allow us to bypass the computation of these norms.

Recall that if

$$\mathbf{D} = (d_{\ell,j}; \ell, j = 1, \cdots, m) \in \mathbb{C}^{m\times m}$$

then the Frobenius norm of $\mathbf{D}$ is

$$\|\mathbf{D}\|_F := \left(\sum_{\ell=1}^{m} \sum_{j=1}^{m} |d_{\ell,j}|^2 \right)^{1/2}.$$

Note that this is not an induced matrix norm. However, since

$$\|\mathbf{D}\| \leq \|\mathbf{D}\|_F \leq \sqrt{m}\|\mathbf{D}\| \tag{6}$$

both norms are equivalent. The Frobenius norm is clearly easier to estimate. We have

Proposition 1. *Let $\mathbf{h} = \{h_1, \cdots, h_m\}$, $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_r\}$, assume that $T(\mathbf{h})$ is a Riesz sequence in $L^2(\mathbb{R}^d)$, that $\boldsymbol{\psi}$ is a set of functions such that $S(\boldsymbol{\psi}) \subset S(\mathbf{h})$, and let $\mathbf{Q}(\mathbf{x})$ denote the transition matrix from $T(\mathbf{h})$ to $T(\boldsymbol{\psi})$. Then $T(\boldsymbol{\psi})$ is a Riesz basis of $S(\mathbf{h})$ if $r = m$ and both $||\mathbf{Q}(\mathbf{x})||_F$ and $(\det[\mathbf{Q}(\mathbf{x})])^{-1}$ are in $L_\infty(\mathbb{T}^d)$. If, moreover, $\mathbf{Q}(\mathbf{x})$ is continuous on $\mathbb{T}^d$, then $T(\boldsymbol{\psi})$ is a Riesz basis of $S(\mathbf{h})$ if and only if $r = m$ and $\mathbf{Q}(\mathbf{x})$ is nonsingular for every $\mathbf{x} \in \mathbb{T}^d$.*

Proof. The assertion follows from [4, Theorem 2.26(iv)], (6), and the identity

$$\mathbf{Q}^{-1}(\mathbf{x}) = \operatorname{Adj} \mathbf{Q}(\mathbf{x}) / \det \mathbf{Q}(\mathbf{x}).$$

□

Note that, except for a special case, [4, Theorem 2.26(iv)] and Proposition 1 only give sufficient conditions. However, if $T(\mathbf{h})$ is an orthonormal sequence we can do better.

Theorem 5. *Let $\mathbf{h} = \{h_1, \cdots, h_m\}$, $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\}$, assume that $T(\mathbf{h})$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, that $\boldsymbol{\psi}$ is a set of functions such that $S(\boldsymbol{\psi}) \subset S(\mathbf{h})$, and let $\mathbf{Q}(\mathbf{x})$ denote the transition matrix from $T(\mathbf{h})$ to $T(\boldsymbol{\psi})$. Then*

$$G_{\boldsymbol{\psi}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) \left(\mathbf{Q}(\mathbf{x})\right)^* \quad a.e., \tag{7}$$

and the following statements are equivalent:

(a) *$T(\boldsymbol{\psi})$ is a Riesz basis of $S(\mathbf{h})$ with bounds $0 < A \leq B$.*
(b) *$r = m$ and for almost every $\mathbf{x} \in \mathbb{T}^d$*

$$||G_{\boldsymbol{\psi}}(\mathbf{x})|| \leq B \quad \text{and} \quad ||G_{\boldsymbol{\psi}}^{-1}(\mathbf{x})|| \leq A^{-1}. \tag{8}$$

(c) *$r = m$ and for almost every $\mathbf{x} \in \mathbb{T}^d$*

$$||\mathbf{Q}(\mathbf{x})|| \leq B^{1/2} \quad \text{and} \quad ||\mathbf{Q}^{-1}(\mathbf{x})|| \leq A^{-1/2}. \tag{9}$$

In particular, $T(\boldsymbol{\psi})$ is an orthonormal basis of $S(\mathbf{h})$ if and only if $r = m$ and $G_{\boldsymbol{\psi}}(\mathbf{x}) = \mathbf{I}$ for almost every $\mathbf{x} \in \mathbb{T}^d$, or, equivalently, if and only if $r = m$ and $\mathbf{Q}(\mathbf{x})$ is a unitary matrix for almost every $\mathbf{x} \in \mathbb{T}^d$.

Proof. From Lemma E we know that $G_{\mathbf{h}}(\mathbf{x}) = \mathbf{I}$. Thus (7) follows from (5).

The equivalence of (a) and (b) follows from [25, Theorem 2.3.6] and Theorem 1.

To prove the equivalence of (b) and (c) we proceed as follows: given a matrix $\mathbf{A}$, let $\rho(\mathbf{A})$ denote its spectral radius. Since

$$||\mathbf{A}|| = \sqrt{\rho(\mathbf{A}^*\mathbf{A})}$$

(Isaacson and Keller [18]), we have

$$||\mathbf{Q}(\mathbf{x})|| = \sqrt{\rho(\mathbf{Q}(\mathbf{x})\mathbf{Q}^*(\mathbf{x}))} = \sqrt{\rho(G_{\boldsymbol{\psi}}(\mathbf{x}))} = \sqrt{||G_{\boldsymbol{\psi}}(\mathbf{x})||} \quad a.e.,$$

and a similar argument establishes the identity

$$||\mathbf{Q}^{-1}(\mathbf{x})|| = \sqrt{||G_{\boldsymbol{\psi}}^{-1}(\mathbf{x})||} \quad a.e.$$

The particular case is a consequence of Lemma D and (7). □

An alternative way to establish the two preceding displayed equations is by using the identity $||\mathbf{A}^{1/2}|| = ||\mathbf{A}||^{1/2}$, which is valid for positive bounded self–adjoint operators, and Theorem 12.34 in Rudin [26].

In [24] Olson and the author proved that if $u \in L^2(\mathbb{R})$ then $T(u)$ cannot be a Riesz basis of $L^2(\mathbb{R})$. As shown by Christensen, Deng and Heil in [11], $T(u)$ cannot even be a frame. This result was improved by Deng and Heil in [15, Theorem 5.1], where they showed that if a function g lies in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then no set of translates of g can be a Schauder basis for $L^2(\mathbb{R})$. We end this section by presenting a further contribution to this problem. Going back to the pertinent definitions in Section 2, recall that the terms "basis generator" and "Schauder basis generator" mean different things. We have not discussed how they are related; this will be partially clarified in the proof of Corollary 7 below.

Theorem 6. *No finite set of functions in $L^2(\mathbb{R}^d)$ may be a basis generator of $L^2(\mathbb{R}^d)$.*

Proof. Assume there is a set $\mathbf{u} = \{u_1, \cdots, u_k\}$ such that $T(\mathbf{u})$ is a basis of $L^2(\mathbb{R}^d)$. Let $\{V_j; j \in \mathbb{Z}\}$ be any multiresolution analysis of multiplicity 1 of $L^2(\mathbb{R}^d)$, let $|a|^r > k$, and let $\mathbf{v} = \{v_1, \cdots, v_m\}$ be a Riesz basis generator of V_r. This implies that $\det[G_{\mathbf{v}}(\mathbf{x})] \neq 0$ for almost every $\mathbf{x} \in \mathbb{T}^d$. Moreover, from Theorem 4 we know that $m = |a|^r > k$. But [4, Theorem 3.12(b)] implies that $len\ S(\mathbf{v}) = m$. Since $len\ L^2(\mathbb{R}^d) \leq k$, we see that $m > len\ L^2(\mathbb{R}^d)$. Thus [5, Theorem 3.12(a)] implies that $\det[G_{\mathbf{v}}(\mathbf{x})]$ vanishes a.e., and we have a contradiction. □

Corollary 7. *No finite set of functions in $L^2(\mathbb{R}^d)$ may be a Schauder basis generator of $L^2(\mathbb{R}^d)$.*

Proof. Assume there is a set $\mathbf{h} = \{h_1, \cdots, h_k\}$ such that $T(\mathbf{h})$ is a Schauder basis of $L^2(\mathbb{R}^d)$, and let $\mathbf{u} = \{u_1, \cdots, u_k\}$ be an orthonormal sequence in $L^2(\mathbb{R}^d)$. Then $S(\mathbf{u}) \subset S(\mathbf{h})$, and Theorem F implies that the transition matrix $Q(\mathbf{x})$ from $T(\mathbf{h})$ to $T(\mathbf{u})$ exists. Applying (5) and Lemma D we therefore conclude that, for almost every $\mathbf{x} \in \mathbb{T}^d$,

$$\mathbf{I} = \mathbf{Q}(\mathbf{x}) G_{\mathbf{h}}(\mathbf{x}) \mathbf{Q}^*(\mathbf{x}).$$

Thus $G_{\mathbf{h}}(\mathbf{x})$ is a.e. nonsingular, which means that $\mathbf{h}$ is a basis generator of $L^2(\mathbb{R}^d)$. This contradicts Theorem 6, and the assertion follows. □

4. Wavelets

In what follows we will assume that $\mathbf{A}$ is a fixed dilation matrix preserving the lattice $\mathbb{Z}^d$.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ will be called an orthonormal, frame, or Riesz wavelet system if the affine sequence

$$\bigcup_{j \in \mathbb{Z}} T(\mathbf{A}^j; \boldsymbol{\psi}) = \{D_j^{\mathbf{A}} T_{\mathbf{k}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is respectively an orthonormal basis, a frame, or a Riesz basis of $L^2(\mathbb{R}^d)$. If $d = 1$ we omit the word "system". If we need to emphasize the connection with the matrix $\mathbf{A}$, we will say that the wavelet system is *generated* by $\mathbf{A}$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be a Riesz wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$; for $j \in \mathbb{Z}$ we define $P_j := S(\mathbf{A}^j; \boldsymbol{\psi})$ and $V_j := \sum_{r<j} P_r$, i.e.,

$$V_j = \sum_{r<j} S(\mathbf{A}^r; \boldsymbol{\psi}).$$

Note that $\boldsymbol{\psi} \in V_1$. Following e.g. Wang [28] and Baggett et al. [7] we say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\boldsymbol{\psi}$ is associated with M. Let W_j denote the orthogonal complement of V_j in V_{j+1}. Then $\boldsymbol{\psi}$ is an orthonormal wavelet associated with M if and only if $P_j = W_j$ for every $j \in \mathbb{Z}$, and $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

A Riesz wavelet system $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\} \subset L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$ is called *semiorthogonal* if $j \neq q$ implies that $\langle D_j^{\mathbf{A}} T_{\mathbf{k}} \psi_\ell, D_q^{\mathbf{A}} T_{\mathbf{n}} \psi_r \rangle = 0$ for every $\mathbf{k}, \mathbf{n} \in \mathbb{Z}^d$ and $\ell, r = 1, \cdots, m$. It follows that a finite set $\boldsymbol{\psi} \subset L^2(\mathbb{R}^d)$ is a semiorthogonal Riesz basis associated with an MRA if and only if $P_j = W_j$ for every $j \in \mathbb{Z}$ and $T(\boldsymbol{\psi})$ is a Riesz basis of W_0. In [29] Weiss and Wilson showed that if $\mathbf{A} = 2\mathbf{I}$ then any orthonormal wavelet system associated with an MRA of multiplicity 1 has exactly $2^d - 1$ functions. A generalization of

this assertion is implicit in the work of Bownik and Garrigós [3].

Theorem H. *If $\boldsymbol{\psi}$ is a Riesz wavelet system associated with a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and $a := \det \mathbf{A}$, then it has exactly $(|a|-1)n$ functions.*

The proof of this theorem follows by noting that, in view of [3, Proposition 2.4], the value D that appears in the proof of [3, Theorem 3.3] is equal to n.

This result was established for orthonormal wavelet systems by Cabrelli and Gordillo in [8], although it is not stated as a formal theorem. For semiorthogonal Riesz wavelet systems it follows from Corollary 3, and for biorthogonal Riesz wavelet systems it follows from a result of Calogero and Garrigós ([10, Theorem 2.11]).

Now we turn to the problem of proving the existence of an othonormal wavelet system associated with a given multiresolution analysis M generated by a matrix $\mathbf{A} \in \mathbb{Z}^{d\times d}$. The first result of this nature, for $\mathbf{A} = 2\mathbf{I}$, was proved by Gröchenig [16] (this proof can also be found in [22, 23]). See also Jia and Shen [19, Theorem 3.7]. In e.g. Soardi [27] and Wojtaszczyk [30] it is shown that every multiresolution analysis of multiplicity 1 in $L^2(\mathbb{R}^d)$ has an orthonormal wavelet system associated with it. (For work on a different aspect of this problem, see [1, 12]). Applying Corollary 3 we have the following generalization of these results.

Theorem 7. *Every multiresolution analysis of multiplicity n in $L^2(\mathbb{R}^d)$ generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ has an orthonormal wavelet system associated with it, and every orthonormal wavelet system associated with a multiresolution analysis of multiplicity n in $L^2(\mathbb{R}^d)$ generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ has exactly $(|a|-1)n$ functions, where $a := \det \mathbf{A}$.*

We now obtain a representation theorem similar to those obtained by Cabrelli and Gordillo [8, Theorem 4.2] and Cabrelli and Heil [9, Theorem 4.11].

Theorem 8. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and having scaling functions $u_1, \cdots, u_n$. Let $a := \det \mathbf{A}$, $m := |a|n$, and let $\{v_1, \cdots, v_m\}$ be an orthonormal basis generator of V_1. The following propositions are equivalent:*

(a) *$\{w_1, \cdots w_{m-n}\}$ is an orthonormal wavelet system associated with M.*

(b) *There is an $m \times m$ matrix $\mathbf{Q}(\mathbf{x})$ of $\mathbb{Z}^d$–periodic and measurable functions and a.e. unitary on T^d such that, if*

$$(\widehat{y_1}(\mathbf{x}), \cdots \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})(\widehat{v_1}(\mathbf{x}), \cdots \widehat{v_m}(\mathbf{x}))^T,$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n$$

and

$$y_{(\ell-1)|a|+k} = w_{(\ell-1)|a|+k-\ell}; \quad 1 \le \ell \le n, \quad 2 \le k \le |a|.$$

Proof. The proof is obtained by noting that $\{w_1, \cdots, w_{m-n}\}$ is an orthonormal wavelet system if and only if $T(w_1, \cdots w_{m-n})$ is an orthonormal basis of W_0. Since $V_1 = W_0 \oplus V_0$ and $\mathbf{u}$ is an orthonormal basis generator of V_0, this condition in turn is satisfied if and only if

$$T(u_1, w_1, \cdots, w_{|a|-1}, \cdots, u_n, w_{m-n-|a|+2}, \cdots, w_{m-n})$$

is an orthonormal basis generator of V_1. The assertion now follows from Theorem 5. □

Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$. By orthogonality we know that

$$V_1 = S(A, u_1) \oplus S(A, u_2) \oplus \cdots \oplus S(A, u_n).$$

Theorem 3 implies that there are functions $v_{\ell,k}$ such that

$$\{v_{\ell,1}, \cdots, v_{\ell,|a|}\} \tag{10}$$

is an orthonormal basis generator of $S(\mathbf{A}, u_\ell)$. It follows that

$$\{v_{\ell,k}; 1 \le \ell \le n,\ 1 \le k \le |a|\}$$

is an orthonormal basis generator of V_1.

For $k > 1$ let $\operatorname{diag}\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{k}}$ denote the $k \times k$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. The following proposition was established by Jia and Shen in the discussion that follows the proof of [19, Lemma 3.3] (we adopt the convention that $\operatorname{Arg} 0 = 0$).

Lemma I. *Let* $\mathbf{b} = (b_1, \cdots, b_k)^T \in \mathbb{C}^k$ *be unimodular,* $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^k$, $\omega := Arg\, b_1$ *and* $\mathbf{q} := \mathbf{b} + e^{i\omega}\mathbf{e}$. *Then the matrix*

$$\mathbf{Q} := \operatorname{diag}\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{k}} \left[\overline{\mathbf{I} - 2\mathbf{q}\mathbf{q}^*/\mathbf{q}^*\mathbf{q}}\right]$$

is unitary and has $(b_1, \cdots, b_k)$ *as its first row.*

We can now prove the following theorem which, combined with Theorem 3, provides an explicit closed form representation in Fourier space for orthonormal wavelet systems associated with a given multiresolution analysis of multiplicity n.

Theorem 9. *Let* $M := \{V_j; j \in \mathbb{Z}\}$ *be a multiresolution analysis of multiplicity* n *with scaling functions* $\mathbf{u} := \{u_1, \cdots, u_n\}$, *generated by a matrix* $\mathbf{A}$ *that preserves the lattice* $\mathbb{Z}^d$. *For* $1 \le \ell \le n$, *let (10) be an orthonormal basis generator of* $S(\mathbf{A}, u_\ell)$, *let* $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^{|a|}$, *and*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{|a|} b_{\ell,j}(\mathbf{x})\widehat{v_{\ell,j}}(\mathbf{x}). \tag{11}$$

Define

$$\mathbf{b}_\ell(\mathbf{x}) := (b_{\ell,1}(\mathbf{x}), \cdots, b_{\ell,|a|}(\mathbf{x}))^T, \quad \delta_\ell(\mathbf{x}) := e^{i Arg\, b_{\ell,1}(\mathbf{x})}, \quad \mathbf{q}_\ell(\mathbf{x}) := \mathbf{b}_\ell(\mathbf{x}) + \delta_\ell(\mathbf{x})\mathbf{e},$$
$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_{1,1}(\mathbf{x}), \cdots, \widehat{v}_{1,|a|}(\mathbf{x}), \cdots, \widehat{v}_{n,1}(\mathbf{x}), \cdots, \widehat{v}_{n,|a|}(\mathbf{x}))^T,$$

and

$$\mathbf{Q}_\ell(\mathbf{x}) := \operatorname{diag}\{-\delta_\ell(\mathbf{x}), 1, \cdots, 1\}_{|\mathrm{a}|} \left[\overline{\mathbf{I} - 2\mathbf{q}_\ell(\mathbf{x})\mathbf{q}_\ell(\mathbf{x})^*/\mathbf{q}_\ell(\mathbf{x})^*\mathbf{q}_\ell(\mathbf{x})}\right].$$

Let $a := \det \mathbf{A}$, $m := |a|n$, *and let*

$$\mathbf{Q}(\mathbf{x}) = \Big(q_{\ell,k}(\mathbf{x})\Big)_{\ell,k=1}^{m}$$

be the $m \times m$ *block diagonal matrix*

$$\mathbf{Q}_1(\mathbf{x}) \oplus \mathbf{Q}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{Q}_n(\mathbf{x}) = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_n(\mathbf{x}) \end{pmatrix}.$$

If

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\widehat{\mathbf{v}}(\mathbf{x}),$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n, \tag{12}$$

and

$$\{y_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\}$$

is an orthonormal wavelet system associated with M.

Proof. Since Lemma E implies that

$$\sum_{j=1}^{|a|} |b_{\ell,j}(\mathbf{x})|^2 = 1 \quad a.e.; \quad \ell = 1, \cdots, n,$$

setting

$$\widehat{\mathbf{v}}_\ell(\mathbf{x}) := (\widehat{v}_{\ell,1}(\mathbf{x}), \cdots, \widehat{v}_{\ell,|a|}(\mathbf{x}))^T,$$

and applying Lemma I, we see that

$$(\widehat{y}_{\ell|a|+1}(\mathbf{x}), \cdots \widehat{y}_{\ell|a|+n}(\mathbf{x})^T = \mathbf{Q}_\ell(\mathbf{x})\widehat{\mathbf{v}}_\ell(\mathbf{x}); \quad \ell = 1, \cdots, n,$$

and that $\mathbf{Q}_\ell(\mathbf{x})$ has $(b_{\ell,1}(\mathbf{x}),\cdots,b_{\ell,|a|}(\mathbf{x}))$ as its first row. This implies that for $\ell=1,\cdots,n$

$$q_{(\ell-1)|a|+1,k}=\begin{cases} b_{\ell,k-\ell|a|} & \text{if } (\ell-1)|a|+1\le k<\ell|a| \\ 0 & \text{otherwise}\end{cases}$$

and from (11) we see that (12) follows. Since, moreover, the matrices $\mathbf{Q}_\ell(\mathbf{x})$ are unitary for almost every $\mathbf{x}$ in $\mathbb{T}^d$, it is clear that also $\mathbf{Q}(\mathbf{x})$ is unitary for almost every $\mathbf{x}$ in $\mathbb{T}^d$, and the assertion follows from Theorem 8. □

The proofs of the next two propositions are almost identical to the corresponding univariate proofs in the author's paper [32], and will be omitted.

Proposition 2. *Let $\mathbf{u}:=\{u_1,\cdots,u_m\}\subset L^2(\mathbb{R}^d)$ and let $\boldsymbol{\mu}:=\{\mu_1,\cdots,\mu_m\}$ be a set of measurable $\mathbb{Z}^d$–periodic functions. Let $\boldsymbol{\psi}:=\{\psi_1,\cdots\psi_m\}\subset L^2(\mathbb{R})$ be such that*

$$\widehat{\psi}_\ell(\mathbf{x})=\mu_\ell(\mathbf{x})\widehat{u}_\ell(\mathbf{x})\quad a.e.;\quad \ell=1,\cdots,m. \tag{13}$$

Then:

(a) *If there are constants $B,D>0$ such that $|\mu_\ell(\mathbf{x})|^2\le B$ a.e., $\ell=1,\cdots,m$, and $\mathbf{u}$ is a Bessel wavelet system with bound D, then $\boldsymbol{\psi}$ is a Bessel wavelet system with bound BD.*

(b) *If there are constants $0<A\le B, 0<C\le D$ such that $A\le|\mu_\ell(\mathbf{x})|^2\le B$ a.e., $\ell=1,\cdots,m$, and $\mathbf{u}$ is a frame wavelet system with bounds C and D, then $\boldsymbol{\psi}$ is a frame wavelet system with bounds AC and BD.*

Proposition 3. *Let $\mathbf{u}:=\{u_1,\cdots,u_m\}\subset L^2(\mathbb{R}^d)$ and let $\boldsymbol{\mu}:=\{\mu_1,\cdots,\mu_m\}$ be a set of measurable $\mathbb{Z}^d$–periodic functions. If the functions $\boldsymbol{\psi}$ are defined by (13) and there are constants C and D, $0<C<|D|\sqrt{A/B}$, such that*

$$|\mu_\ell(x)-D|\le C\quad a.e.;\quad \ell=1,\cdots,m,$$

then $\boldsymbol{\psi}$ is a Riesz wavelet system.

The following two theorems generalize the corresponding univariate versions found in [32].

Theorem 10. *Let $M=\{V_j;j\in\mathbb{Z}\}$ be a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$. Let $a:=\det\mathbf{A}$, $m:=(|a|-1)n$, and $\boldsymbol{\psi}:=\{\psi_1,\cdots,\psi_m\}\subset L^2(\mathbb{R}^d)$. If there is an integer $r<0$ such that $\boldsymbol{\psi}\subset V_r$, then $\boldsymbol{\psi}$ cannot be a Riesz wavelet system.*

Proof. If $\boldsymbol{\psi}$ is a Riesz wavelet system then $T(\boldsymbol{\psi})$ is a Riesz sequence in V_r, and Theorems 1 and 4 imply that $(|a|-1)n\le n|a|^r$. Since $(|a|-1)\ge 1$, this is only possible if $r\ge 0$. □

Theorem 11. *Let $M=\{V_j;j\in\mathbb{Z}\}$ be a multiresolution analysis of multiplicity n generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$. Let $a:=\det\mathbf{A}$, $m:=(|a|-1)n$, and $\boldsymbol{\psi}:=\{\psi_1,\cdots,\psi_m\}\subset L^2(\mathbb{R}^d)$. If there is an integer $r\ne 0$ such that $\boldsymbol{\psi}\subset W_r$, then $\boldsymbol{\psi}$ cannot be a Riesz wavelet system.*

Proof. Let $\boldsymbol{\psi}$ be a Riesz wavelet system in W_r. Assume first that $r<0$. Since W_r is then a proper subset of V_0, Theorem 1 implies that $m<n$. Since $(|a|-1)\ge 1$, this is impossible.

Assume now that $r>0$. Since Theorem 4 implies that every Riesz basis generator of W_r must have $m|a|^r$ functions and $m|a|^r>m$, this is also impossible. □

Since $\boldsymbol{\psi}$ is an orthonormal wavelet system (resp. a semiorthogonal Riesz wavelet system) associated with an MRA if and only if $T(\boldsymbol{\psi})$ is an orthonormal basis (resp. Riesz basis) of W_0, Theorem H and Theorem 5 yield the following theorem which, combined with Theorem 9, provides an explicit closed form representation in Fourier space for semiorthogonal wavelet systems associated with a given multiresolution analysis of multiplicity n.

Theorem 12. *Assume that $\mathbf{h}=\{h_1,\cdots,h_m\}$ is an orthonormal wavelet system in $L^2(\mathbb{R}^d)$ associated with a multiresolution analysis M of multiplicity n generated by a dilation matrix matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$, that $\boldsymbol{\psi}=\{\psi_1,\cdots,\psi_r\}$ is a set of functions such that $S(\boldsymbol{\psi})\subset S(\mathbf{h})$, let $a:=\det\mathbf{A}$, and let $\mathbf{Q}(\mathbf{x})$ denote the transition matrix from $T(\mathbf{h})$ to $T(\boldsymbol{\psi})$. Then the following statements are equivalent:*

(a) *$\boldsymbol{\psi}$ is a semiorthogonal Riesz wavelet system with bounds $0<A\le B$, associated with M.*

(b) $r = m = (|a|-1)n$ *and (8) is satisfied for almost every* $\mathbf{x} \in \mathbb{T}^d$.
(c) $r = m = (|a|-1)n$ *and (9) is satisfied for almost every* $\mathbf{x} \in \mathbb{T}^d$.

If, moreover, $\mathbf{Q}(\mathbf{x})$ *is continuous on* $\mathbb{T}^d$*, then* $\boldsymbol{\psi}$ *is a semiorthogonal Riesz wavelet system associated with* M *if and only if* $r = m = (|a|-1)n$ *and* $\mathbf{Q}(\mathbf{x})$ *is nonsingular for every* $\mathbf{x} \in \mathbb{T}^d$*. In particular,* $\boldsymbol{\psi}$ *is an orthonormal wavelet system associated with* M *if and only if* $r = m = (|a|-1)n$ *and* $\mathbf{Q}(\mathbf{x})$ *is a unitary matrix for almost every* $\mathbf{x} \in \mathbb{T}^d$*, or, equivalently, if and only if* $r = m = (|a|-1)n$ *and* $G_{\boldsymbol{\psi}}(\mathbf{x}) = \mathbf{I}$ *for almost every* $\mathbf{x} \in \mathbb{T}^d$.

Example. We will now verify Theorems 8 and 9 for the univariate case.

Let M be a multiresolution analysis in $L^2(\mathbb{R})$ generated by binary dilations, having a scaling function $u(x)$ and associated low pass filter $p(x)$.

From Theorem 3 we deduce that a basis $\{v_1, v_2\}$ of V_1 is given by $v_1(t) = \sqrt{2}u(2t)$ and $v_2(t) = \sqrt{2}u(2t+1)$, whence

$$\widehat{v_1}(x) = 2^{-1/2}\widehat{u}(x/2) \quad \text{and} \quad \widehat{v_2}(x) = 2^{-1/2}e^{\pi i x}\widehat{u}(x/2).$$

Since $\widehat{u}(x) = p(x/2)\widehat{u}(x/2)$, we see that

$$\widehat{u}(x) = \sqrt{2}[p(x/2) + p((x+1)/2)]\widehat{v_1}(x) - \sqrt{2}p((x+1)/2)e^{-\pi i x}\widehat{v_2}(x),$$

and that $\widehat{u}(x) = \sqrt{2}e^{-\pi i x}p(x/2)\widehat{v_2}(x)$. Adding both equations, we conclude that

$$\widehat{u}(x) = b_1(x)\widehat{v_1}(x) + b_2(x)\widehat{v_2}(x), \tag{14}$$

where $b_1(x) := 2^{-1/2}[p(x/2) + p((x+1)/2)]$ and $b_2(x) := 2^{-1/2}[p(x/2) - p((x+1)/2)]e^{-\pi i x}$ are $\mathbb{Z}$–periodic.

Rewriting (1) we have

$$\widehat{\psi}(x) = 2^{-1/2}\nu(x)[\overline{p((x+1)/2)} - \overline{p(x/2)}]e^{\pi i x}\widehat{v_1}(x) + 2^{-1/2}\nu(x)[\overline{p(x/2)} + \overline{p((x+1)/2)}]\widehat{v_2}(x),$$

whence

$$\widehat{\psi}(x) = \nu(x)[-\overline{b_2(x)}\widehat{v_1}(x) + \overline{b_1(x)}\widehat{v_2}(x)] \tag{15}$$

Thus,

$$(\widehat{u}(x), \widehat{\psi}(x))^T = Q(x)(\widehat{v_1}(x), \widehat{v_2}(x))^T,$$

where

$$Q(x) = \begin{pmatrix} b_1(x) & , & b_2(x) \\ -\overline{b_2(x)} & , & \overline{b_1(x)} \end{pmatrix}.$$

From Lemma E we conclude that $Q(x)$ is unitary, and we have verified Theorem 8 for this particular case.

Let us now turn to Theorem 9. Let $q^*(x) := (q(x))^*$ and $\delta(x) := e^{i\operatorname{Arg} b_1(x)}$; since

$$q(x) = (b_1(x) + \delta(x), b_2(t))^T,$$

we see that

$$q^*(x)q(x) = (1 + |b_1(x)|)^2 + |b_2(x)|^2 = 2(|b_1(x)| + 1)$$

and

$$q(x)q^*(x) = \begin{pmatrix} [1 + |b_1(x)|]^2 & , & [b_1(x) + \delta(x)]\overline{b_2(x)} \\ \overline{[b_1(x) + \delta(x)]}b_2(x) & , & |b_2(x)|^2 \end{pmatrix}.$$

Thus

$$q^*(x)q(x)\mathbf{I} - 2q(x)q^*(x) = \begin{pmatrix} -2|b_1(x)| - 2|b_1(x)|^2 & , & -2[b_1(x) + \delta(x)]\overline{b_2(x)} \\ -2\overline{[b_1(x) + \delta(x)]}b_2(x) & , & 2|b_1(x)| + 2 - |b_2(x)|^2 \end{pmatrix}.$$

Since $|b_2(x)|^2 = 1 - |b_1(x)|^2$, we see that $2|b_1(x)| + 2 - |b_2(x)|^2 = 2|b_1(x)|(|b_1(x)| + 1)$. Therefore

$$Q(x) = [q(x)q^*(x)]^{-1} \begin{pmatrix} -\delta(x) & , & 0 \\ & & \\ 0 & , & 0 \end{pmatrix} \overline{[q^*(x)q(x)\mathbf{I} - 2q(x)q^*(x)]}$$

$$= [q(x)q^*(x)]^{-1} \begin{pmatrix} 2\delta(x)|b_1(x)|(|b_1(x)|+1) & , & 2\delta(x)\overline{[b_1(x)+\delta(x)]}b_2(x) \\ & & \\ -2(b_1(x)+\delta(x)\overline{b_2(x)} & , & 2|b_1(x)|(|b_1(x)|+1) \end{pmatrix}.$$

Setting $V(x) := (\widehat{v_1}(x), \widehat{v_2}(x))^T$, we see that $Q(x)V(x) = (\widehat{r_1}(x), \widehat{r_2}(x))^T$, where

$$\widehat{r_1}(x) = [2(|b_1(x)|+1)]^{-1} \left\{ 2\delta(x)|b_1(x)|(|b_1(x)|+1)\widehat{v_1}(x) + 2\delta(x)\overline{[b_1(x)+\delta(x)]}b_2(x)\widehat{v_2}(x) \right\}$$

$$= b_1(x)\widehat{v_1}(x) + b_2(x)\widehat{v_2}(x)] = u(x)$$

(the last identity follows from (14)) and

$$\widehat{r_2}(x) = [2(|b_1(x)|+1)]^{-1} \left\{ -2(b_1(x)+\delta(x))\overline{b_2(x)}\widehat{v_1}(x) + 2|b_1(x)|(|b_1(x)|+1)\widehat{v_2}(x) \right\}$$

$$= \delta(x)[-\overline{b_2(x)}\widehat{v_1}(x) + \overline{b_1(x)}\widehat{v_2}(x)].$$

Since $\widehat{r_2}(x)$ is of the form (15) we have verified that it is indeed an orthonormal wavelet associated with the multiresolution analysis M.

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