MATRIX–VALUED WAVELETS AND MULTIRESOLUTION ANALYSIS

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Abstract. We introduce the notions of matrix–valued wavelet set and matrix–valued multiresolution analysis (A-MMRA) associated with a fixed dilation given by an expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 1$ such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, in a matrix–valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $n \geq 1$. These are generalizations of the corresponding notions defined by Xia and Suter in 1996 for the case where $d = 1$ and A is the dyadic dilation. We show several properties of orthonormal sequences of translates by integers of matrix–valued functions, focusing on those related to the structure of A-MMRA's and their connection with matrix–valued wavelet sets. Further, we present a strategy for constructing matrix–valued wavelet sets from a given A-MMRA and, in addition, we characterize those matrix–valued wavelet sets which may be built from an A-MMRA.

1. Introduction

Given a fixed expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 1$, such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, we introduce the notion of matrix–valued wavelet and matrix–valued multiresoltuion analysis associated to A in a matrix–valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $n \geq 1$. A linear map A is said to be expansive if all (complex) eigenvalues of A have modulus greater than 1. The subject of this paper is the study of such wavelets and multiresolution analyses. Our starting point is the paper by Xia and Suter [22] where the notion of matrix–valued wavelet and matrix–valued multiresolution analysis have been introduced and studied for the case of $d = 1$ and dyadic dilations. Subsequently, and in this particular context, there appeared several papers related to matrix–valued multiresolution analyses and matrix–valued wavelets and their construction, e.g. [25], [1], [23], [28]. The notion of matrix–valued multiresolution analysis and matrix–valued wavelets when $d = 1$ and A may be any arbitrary integer dilation were introduced in [6], where a necessary and sufficient condition for the existence of matrix–valued wavelets and an algorithm for constructing compactly supported matrix–valued wavelets associated with an integer dilation factor m are presented. For the case $m = 4$ see [4].

Relaxing requirements, the articles [21], [5], [11], [8] study biorthogonal matrix–valued wavelets where $d = 1$ and A is the dyadic dilation.

Since matrix–valued function spaces $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ are related to video imaging, we generalize results in [27] to these spaces with the purpose of showing that the ideas developed there for scalar–valued wavelets and multiresolution analysis fit perfectly in this context. That is our motivation for writing this article.

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This work is organized as follows. In Section 2 we present the definitions and notation that will be used. Section 3 contains several properties of orthonormal sequences of integer translates of a function in a matrix–valued space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, focusing on those related to multiresolution analyses and their connection with wavelet sets. Section 4 is devoted to the study of matrix–valued wavelet sets in a signal space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, associated with a dilation given by an expansive linear map A. In addition, as a method for constructing these matrix–valued wavelet sets we introduce the notion of vector–valued multiresolution analysis associated with an expansive linear map A (A-MMRA). Futher, we study the structure of A-MMRA's, present a strategy for constructing matrix–valued wavelet sets and characterize those sets constructed from a given A-MMRA. Our results are given in the context of Fourier space.

2. Notation and basic definitions

The sets of integers, real and complex numbers will be denoted by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. The d–fold product of the interval $[0,1)$ with itself will be denoted $\mathbb{T}^d$. Thus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 1$.

Unless otherwise indicated, I_n, $n \geq 1$, will denote the $n\times n$ identity matrix and $\mathbf{0}_n$ will denote the $n \times n$ null matrix.

Given an $n\times n$, $n \geq 1$, complex matrix M, $a_{ml} \in \mathbb{C}$ will denote the element on the m-th row and the l-th column of M. The complex vector space of all $n \times n$ complex matrices M will be denoted by $\mathcal{M}_n(\mathbb{C})$. Recall that a matrix $M \in \mathcal{M}_n(\mathbb{C})$ is said to be unitary if $MM^* = I_n$ where M^* is the transpose of the complex conjugate of M.

Let

$$l^2(\mathbb{N}, \mathbb{C}^{n\times n}) := \{\mathbf{M} = \{M_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_n(\mathbb{C}) : \|\mathbf{M}\| = (\sum_{m,l=1}^{n} \sum_{k\in\mathbb{N}} |a_{ml}(k)|^2)^{1/2} < \infty\}.$$

The space $l^2(\mathbb{Z}^d, \mathbb{C}^{n\times n})$ is similarly defined.

All functions considered in this paper will be assumed to be measurable.

Given $d, n \geq 1$, by $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ we will denote the space

$$\{\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2n}(\mathbf{x}) \\ & & \cdots & \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \cdots & f_{nn}(\mathbf{x}) \end{pmatrix} : f_{ml} \in L^2(\mathbb{R}^d), m, l = 1, ..., n\}.$$

We will also write $\mathbf{f}(\mathbf{x}) = (f_{ml}(\mathbf{x}))_{m,l=1,\dots,n}$. The spaces $L^p(E, \mathbb{C}^{n\times n})$, $1 \leq p < \infty$, where E is a measurable set in $\mathbb{R}^d$ are defined similarly by replacing $\mathbb{R}^d$ and 2 by the E and p respectively. If we write $\mathbf{f} \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$ we will also mean that $\mathbf{f}$ is defined on the whole space $\mathbb{R}^d$ as a $\mathbb{Z}^d$-periodic matrix–valued function.

Given $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $\|\mathbf{f}\|$, will denote the Frobenius norm defined by (see [22])

$$\|\mathbf{f}\| := (\sum_{m,l=1}^{n} \int_{\mathbb{R}^d} |f_{ml}(\mathbf{x})|^2 d\mathbf{x})^{1/2}. \tag{1}$$

The integral of a matrix–valued function $\mathbf{f}$, $\int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})d\mathbf{x}$, is defined by

$$\int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})d\mathbf{x} := (\int_{\mathbb{R}^d} f_{ml}(\mathbf{x})d\mathbf{x})_{m,l=1,\dots,n}.$$

The Fourier transform of a matrix–valued function f will be denoted by $\widehat{f}$. For $f \in L^1(\mathbb{R}^d, \mathbb{C}^{n\times n}) \cap L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$

$$\widehat{\mathbf{f}}(\mathbf{t}) := \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{t}}d\mathbf{x}.$$

For two matrix-valued functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$,

$$\langle \mathbf{f}, \mathbf{g}\rangle := \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})\mathbf{g}^*(\mathbf{x})d\mathbf{x} \tag{2}$$

and

$$[\mathbf{f}, \mathbf{g}](\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{f}(\mathbf{t}+\mathbf{k})\mathbf{g}^*(\mathbf{t}+\mathbf{k}).$$

Note that $\langle \cdot, \cdot\rangle$ is matrix–valued and therefore it is not an inner product. It has the following properties:

(a) For every $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$,

$$\langle \mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{g}, \mathbf{f}\rangle^*;$$

(b) For every $\mathbf{f}, \mathbf{g}, \mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ and every $M_1, M_2 \in \mathcal{M}_n(\mathbb{C})$,

$$\langle M_1\mathbf{f} + M_2\mathbf{h}, \mathbf{g}\rangle = M_1 \langle \mathbf{f}, \mathbf{g}\rangle + M_2 \langle \mathbf{h}, \mathbf{g}\rangle.$$

Moreover, the scalar Plancherel formula implies that also in the matrix–valued case

$$\langle \mathbf{f}, \mathbf{g}\rangle = \left\langle \widehat{\mathbf{f}}, \widehat{\mathbf{g}}\right\rangle.$$

It is also readily seen that

$$\|\mathbf{f}\| = (\text{trace } \langle \mathbf{f}, \mathbf{f}\rangle)^{1/2}.$$

Given an invertible map $M : \mathbb{R}^n \to \mathbb{R}^n$, for every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $\mathbf{D}_j^M$ and the translation operator $\mathbf{T}_\mathbf{k}$ are defined on $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ by

$$\mathbf{D}_j^M\mathbf{f}(\mathbf{t}) := d_M^{j/2}\mathbf{f}(M^j\mathbf{t}) \quad \text{and} \quad \mathbf{T}_\mathbf{k}\mathbf{f}(\mathbf{t}) := \mathbf{f}(\mathbf{t}+\mathbf{k}),$$

where $d_M = |\det M|$. A set $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is called *shift-invariant* if $\mathbf{f} \in S$ implies that $\mathbf{T}_\mathbf{k}\mathbf{f} \in S$ for every $\mathbf{k} \in \mathbb{Z}^n$. Let $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, then

$$\mathbf{T}(\mathbf{F}) := \{\mathbf{T}_\mathbf{k}\mathbf{f} \ : \ \mathbf{f} \in \mathbf{F}, \mathbf{k} \in \mathbb{Z}^n\} \quad \text{and} \quad S(\mathbf{F}) := \overline{\text{span}}\mathbf{T}(\mathbf{F}),$$

where the closure is in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ then $S(\mathbf{F})$ is called a *finitely generated shift-invariant space* or FSI and the functions $\mathbf{f}_l$, $l = 1, ..., m$ are called the generators of $S(\mathbf{F})$. In this case we will also use the symbols $\mathbf{T}(\mathbf{f}_1, ..., \mathbf{f}_m)$ and $S(\mathbf{f}_1, ..., \mathbf{f}_m)$ to denote $\mathbf{T}(\mathbf{F})$ and $S(\mathbf{F})$ respectively. If $\mathbf{F}$ contains a single element, then $S(\mathbf{F})$ is called a *principal shift-invariant space* or PSI.

Two functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ are said to be orthogonal if $\langle \mathbf{f}, \mathbf{g}\rangle = \mathbf{0}_n$. Further, let V, W be two closed subspaces in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $W \subset V$, then the *orthogonal complement* of W in V is the closed subspace defined by

$$W^\perp = \{\mathbf{g} \in V : \langle \mathbf{g}, \mathbf{f}\rangle = \mathbf{0}_n \quad \forall \mathbf{f} \in W\}.$$

A sequence $\{\mathbf{f}_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is called an orthonormal set in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ if

$$\langle \mathbf{f}_k, \mathbf{f}_l \rangle = \begin{cases} I_n & \text{if } k = l \\ \mathbf{0}_n & \text{if } k \neq l. \end{cases} \tag{3}$$

Given a closed subspace S in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, a sequence $\{\mathbf{f}_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is called an *orthonormal basis* for S if it satisfies (3), and moreover, for any $\mathbf{g} \in S$ there exists an unique sequence of constant matrices $\{H_k\}_{k=1}^{\infty} \in l^2(\mathbb{N}, \mathbb{C}^{n\times n})$ such that

$$\mathbf{g}(\mathbf{x}) = \sum_{k=1}^{\infty} H_k \mathbf{f}_k(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \mathbb{R}^d$$

where, for each $\mathbf{x}$, $H_k\mathbf{f}_k(\mathbf{x})$ is the product of the $n \times n$ matrices H_k and $\mathbf{f}_k(\mathbf{x})$, and the convergence for the infinite sum is in the sense of the norm $\|\cdot\|$ defined by (1). It readily follows that for every $k = 1, 2, \dots,$

$$H_k = \langle \mathbf{g}, \mathbf{f}_k \rangle, \qquad \text{and} \qquad \|\{H_k\}_{k=1}^{\infty}\| = \|\mathbf{g}\|. \tag{4}$$

Given a set of matrix–valued functions $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, its Gramian matrix will be denoted by $\mathbf{G}[\mathbf{f}_1, ..., \mathbf{f}_m](\mathbf{t})$ or $\mathbf{G}_{\mathbf{F}}(\mathbf{t})$ and defined as follows:

$$\mathbf{G}_{\mathbf{F}}(\mathbf{t}) := \left([\widehat{\mathbf{f}}_l, \widehat{\mathbf{f}}_j](\mathbf{t})\right)_{l,j=1}^{m}.$$

3. Orthonormal bases of translates

In this section we show several properties on orthonormal sequences of integral translates of functions in a matrix–valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. We focus on those properties closely related to matrix–valued wavelets and matrix–valued multiresoltion analyses, concepts that will be discussed in the next section. Most of the properties presented here are well known in the scalar–valued function space context (cf. e.g. [27]).

The following lemma generalizes a result in [22].

Lemma 1. *Let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. Then $\mathbf{T}(\mathbf{F})$ is an orthonormal sequence in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ if and only if $\mathbf{G}_{\mathbf{F}}(\mathbf{t}) = I_{nm}$ a.e.*

Proof. Let us prove the necessity. By the orthonormality of $\mathbf{T}(\mathbf{F})$, given $j, p \in \{1, ..., m\}$ and $\mathbf{k} \in \mathbb{Z}^d$ we have

$$\int_{\mathbb{R}^d} \mathbf{f}_j(\mathbf{x})\mathbf{f}_p^*(\mathbf{x} - \mathbf{k})d\mathbf{x} = \delta(j,p)\delta(\mathbf{k}, \mathbf{0})I_n, \tag{5}$$

where $\delta(\alpha, \beta) = 1$ if $\alpha = \beta$ and $\delta(\alpha, \beta) = 1$ if $\alpha \neq \beta$. By Plancherel's formula,

$$\begin{aligned} \delta(j,p)\delta(\mathbf{k}, \mathbf{0})I_n &= \int_{\mathbb{R}^d} \widehat{\mathbf{f}}_j(\mathbf{t})\widehat{\mathbf{f}_p}^*(\mathbf{t})e^{2\pi i\mathbf{k}}d\mathbf{t} \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_{[-1/2,1/2]^d+\mathbf{k}} \widehat{\mathbf{f}}_j(\mathbf{t})\widehat{\mathbf{f}_p}^*(\mathbf{t})e^{2\pi i\mathbf{k}}d\mathbf{t} \\ &= \int_{[-1/2,1/2]^d} \sum_{\mathbf{k}\in\mathbb{Z}^d} [\widehat{\mathbf{f}}_j, \widehat{\mathbf{f}}_p](\mathbf{t})e^{2\pi i\mathbf{k}}d\mathbf{t}, \qquad \forall \mathbf{k} \in \mathbb{Z}^d. \end{aligned} \tag{6}$$

This implies that $[\widehat{\mathbf{f}}_j, \widehat{\mathbf{f}}_p](\mathbf{t}) = \delta(j,p)I_n$ a.e. on $\mathbb{R}^d$, whence the assertion follows.

Conversely, note that the orthonormality of $\mathbf{T}(\mathbf{F})$ follows immediately from $\mathbf{G}_{\mathbf{F}}(\mathbf{t}) = I_{nm}$ a.e., (5) and (6). $\square$

Lemma 2. *Let* $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *and assume that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of a closed subspace* $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$*. Then, a matrix–valued function* $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *belongs to* S *if and only if there are* $\mathbb{Z}^d$*-periodic functions* $\mathbf{H}_j \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$*,* $j = 1, ..., m$*, such that*

$$\widehat{\mathbf{g}}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_j(\mathbf{t})\widehat{\mathbf{f}}_j(\mathbf{t}) \qquad \textit{a.e. on } \mathbb{R}^d, \tag{7}$$

and

$$\| \mathbf{g} \|^2 = \sum_{j=1}^{m} \| \mathbf{H}_j \|^2 \tag{8}$$

Proof. Suppose that $\mathbf{g} \in S$, then we may represent it in terms of the orthonormal basis $\mathbf{T}(\mathbf{F})$ as

$$\mathbf{g}(\mathbf{x}) = \sum_{j=1}^{m} \sum_{\mathbf{k}\in\mathbb{Z}^d} H_{j,\mathbf{k}} \mathbf{f}_j(\mathbf{x} - \mathbf{k}), \tag{9}$$

where $\{H_{j,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d, \mathbb{C}^{n\times n})$, $j \in \{1, ..., m\}$, and the convergence of the sum is in the sense of the norm $\|\cdot\|$ defined by (1). Thus, taking the Fourier transform in (9) we obtain (7) with $\mathbf{H}_j(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} H_{j,\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}$.

From (9) and (4) we deduce that $\| g \| = \| \{H_{j,\mathbf{k}}\}_{j=1,...,m,\mathbf{k}\in\mathbb{Z}^d} \|$. Since $\| \mathbf{H}_j \| = \| \{H_{j,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d} \|$, equation (8) follows.

Conversely, assume that (7) holds. Since $\mathbf{H}_j(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} H_{j,\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}$ with $H_{j,\mathbf{k}} \in l^2(\mathbb{Z}^d, \mathbb{C}^{n\times n})$, we deduce that (9) is satisfied in the sense of convergence in norm, and therefore $\mathbf{g} \in S$. $\square$

Lemma 3. *Let* $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ *and* $\mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_p\}$ *be in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$*. Assume that* $\mathbf{T}(\mathbf{G})$ *and* $\mathbf{T}(\mathbf{F})$ *are orthonormal sequences in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$*, and that there are* $\mathbb{Z}^d$*-periodic functions* $\mathbf{H}_{l,j} \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$*,* $j = 1, ..., m$*,* $l = 1, ..., p$*, such that*

$$\widehat{\mathbf{g}_l}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\widehat{\mathbf{f}_j}(\mathbf{t}) \qquad \textit{a.e. on } \mathbb{R}^d \qquad l = 1, ..., p. \tag{10}$$

Then

$$\sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}^*_{r,j}(\mathbf{t}) = I_n\delta(l, r) \qquad \textit{a.e. on } \mathbb{R}^d \qquad l, r \in \{1, ..., p\}. \tag{11}$$

Proof. Since both sequences are orthonormal, given $l, r \in \{1, ..., p\}$, (3) yields

$$\begin{aligned} I_n\delta(l, r) &= [\widehat{\mathbf{g}_l}, \widehat{\mathbf{g}_r}](\mathbf{t}) = [\sum_{j=1}^{m} \mathbf{H}_{l,j}\widehat{\mathbf{f}}_j, \sum_{j=1}^{m} \mathbf{H}_{r,j}\widehat{\mathbf{f}}_j](\mathbf{t}) \\ &= \sum_{j=1}^{m}\sum_{q=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})[\widehat{\mathbf{f}}_j, \widehat{\mathbf{f}}_q](\mathbf{t})\mathbf{H}^*_{r,q}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}^*_{r,j}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d. \end{aligned}$$

$\square$

We are now ready to prove

Proposition 1. *Let $p \le m$ and let S be a closed subspace of $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. Let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ and $\mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_p\}$ be such that $\mathbf{T}(\mathbf{F})$ and $\mathbf{T}(\mathbf{G})$ belong to S. If $\mathbf{T}(\mathbf{F})$ is an orthonormal basis for S, then $\mathbf{T}(\mathbf{G})$ is an orthonormal sequence in S if and only if there exists a matrix $\mathbf{Q}(\mathbf{t}) := \big(h_{q,r}(\mathbf{t})\big)_{q,r=1}^{np,nm}$ where $h_{q,r} \in L^2(\mathbb{T}^d)$, which satisfies $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. $\mathbf{t}$ on $\mathbb{R}^d$ and also,*

$$\big(\widehat{\mathbf{g}}_1(\mathbf{t}), ..., \widehat{\mathbf{g}}_p(\mathbf{t})\big)^T = \mathbf{Q}(\mathbf{t})\big(\widehat{\mathbf{f}}_1(\mathbf{t}), ..., \widehat{\mathbf{f}}_m(\mathbf{t})\big)^T \quad \text{a.e. on } \mathbb{R}^d. \tag{12}$$

The $\mathbb{Z}^d$-periodic matrix

$$\mathbf{Q}(\mathbf{t}) = \big(h_{q,r}(\mathbf{t})\big)_{q,r=1}^{np,nm}$$

will be called a *transition matrix* from the sequence $\mathbf{T}(\mathbf{F})$ to the sequence $\mathbf{T}(\mathbf{G})$.

Proof. To prove the necessity we proceed as follows: Since $\mathbf{T}(\mathbf{F})$ is an orthonormal basis for S and $\mathbf{T}(\mathbf{G}) \subset S$, Lemma 2 tells us that there are $\mathbf{H}_{l,j} \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $j = 1, ..., m$ and $l = 1, ..., p$, such that

$$\widehat{\mathbf{g}_l}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\widehat{\mathbf{f}_j}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d \qquad l = 1, ..., p. \tag{13}$$

Let $\mathbf{Q}(\mathbf{t})$ be the $np \times nm$ block matrix $\mathbf{Q}(\mathbf{t}) := \big(\mathbf{H}_{l,j}(\mathbf{t})\big)_{l,j=1}^{p,m}$, and for $q = 1, \dots np$, let $\mathbf{v}_q(\mathbf{t}) = (h_{q,1}(\mathbf{t}), ..., h_{q,nm}(\mathbf{t}))$, $q = 1, ..., np$, be the q-th row of $\mathbf{Q}(\mathbf{t})$. (Note that every $h_{q,r}$ belongs to $L^2(\mathbb{T}^d)$). Then, (11) implies that the vectors $\{\mathbf{v}_q(\mathbf{t}) : q \in \{1, ..., np\}\}$ are orthonormal a.e.($\mathbf{t}$). Thus, setting $\mathbf{Q}(\mathbf{t}) := (h_{q,r}(\mathbf{t}))_{q,r=1}^{np,nm}$ we conclude that $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$. Finally, note that (13) readily implies (12).

To prove the sufficiency, for any $l \in \{1, ..., p\}$ and $j \in \{1, ..., m\}$ let $\mathbf{H}_{l,j}$ in $L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$ be defined by $\mathbf{H}_{l,j} := \big(h_{q,r}\big)_{q=(l-1)n+1, r=(j-1)n+1}^{ln,jn}$. Then (12) yields (13). In addition, the assumption $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. on $\mathbb{R}^d$ implies that

$$\sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}_{b,j}^*(\mathbf{t}) = I_n\delta(l, b) \qquad \text{a.e. on } \mathbb{R}^d \qquad l, b \in \{1, ..., p\}.$$

We complete the proof by showing that the Gramian associated to $\mathbf{G}$ is the unitary matrix a.e. on $\mathbb{R}^d$ and applying Lemma 1. For $l \in \{1, ..., p\}$ and $b \in \{1, ..., m\}$ we have:

$$\begin{aligned}
[\widehat{\mathbf{g}_l}, \widehat{\mathbf{g}_b}](\mathbf{t}) &= [\sum_{j=1}^{m} \mathbf{H}_{l,j}\widehat{\mathbf{f}_j}, \sum_{j=1}^{m} \mathbf{H}_{b,j}\widehat{\mathbf{f}_j}](\mathbf{t}) \\
&= \sum_{j=1}^{m}\sum_{q=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})[\widehat{\mathbf{f}_j}, \widehat{\mathbf{f}_q}](\mathbf{t})\mathbf{H}_{b,q}^*(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}_{b,j}^*(\mathbf{t}) = I_n\delta(l, b)
\end{aligned}$$

a.e. on $\mathbb{R}^d$, and the assertion follows. □

Proposition 2. *Assume that $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ and $\mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_p\}$ are functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If $\mathbf{T}(\mathbf{F})$ and $\mathbf{T}(\mathbf{G})$ are orthonormal bases of the same closed subspace $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, then $m = p$.*

Proof. By the symmetry in the notation we may assume, without loss of generality, that $p > m$. Since $\mathbf{T}(\mathbf{F})$ is an orthonormal basis for S and $\mathbf{T}(\mathbf{G}) \subset S$, we infer from Proposition 1 that there exists an $np \times nm$ matrix $\mathbf{Q}(\mathbf{t})$ such that $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. This means that the np vectors defined by the rows of the matrix $\mathbf{Q}(\mathbf{t})$

are orthonormal in the complex vector space $\mathbb{C}^{nm}$. Since $nm < np$, we get a contradiction. $\square$

Proposition 3. *Asssume that the matrix–valued functions*

$$\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}, \mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$$

are such that $\mathbf{T}(\mathbf{F})$ *and* $\mathbf{T}(\mathbf{G})$ *are orthonormal sequences in a closed subspace* $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. *If* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis for* S *and there exists a matrix* $\mathbf{Q}(\mathbf{t}) := \big(h_{l,j}(\mathbf{t})\big)_{l,j=1}^{nm}$, *where* $h_{l,j} \in L^2(\mathbb{T}^d)$, *such that* $\mathbf{Q}(\mathbf{t})$ *is unitary a.e.* $(\mathbf{t})$ *on* $\mathbb{R}^d$ *and (12) holds, then* $\mathbf{T}(\mathbf{G})$ *is an orthonormal basis for* S.

Proof. According to Proposition 1, $\mathbf{T}(\mathbf{G})$ is an orthonormal sequence in S. Thus it suffices to show that $S = \overline{\operatorname{span}}\mathbf{T}(\mathbf{G})$. The hypotheses imply that we only need to check that $S \subset \mathbf{T}(\mathbf{G})$. Let $\mathbf{h} \in S$ then, by Lemma 2 there exist $\mathbf{H}_j \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $j = 1, ..., m$, such that

$$\widehat{\mathbf{h}}(\mathbf{t}) = \big(\mathbf{H}_1(\mathbf{t}), ..., \mathbf{H}_m(\mathbf{t})\big)\big(\widehat{\mathbf{f}_1}(\mathbf{t}), ..., \widehat{\mathbf{f}_m}(\mathbf{t})\big)^T \qquad \text{a.e. on } \mathbb{R}^d.$$

Thus, by (12)

$$\begin{aligned}\widehat{\mathbf{h}}(\mathbf{t}) &= \big(\mathbf{H}_1(\mathbf{t}), ..., \mathbf{H}_m(\mathbf{t})\big)\mathbf{Q}^*(\mathbf{t})\big(\widehat{\mathbf{g}_1}(\mathbf{t}), ..., \widehat{\mathbf{g}_m}(\mathbf{t})\big)^T \\ &= \big(\mathbf{L}_1(\mathbf{t}), ..., \mathbf{L}_m(\mathbf{t})\big)\big(\widehat{\mathbf{g}_1}(\mathbf{t}), ..., \widehat{\mathbf{g}_m}(\mathbf{t})\big)^T \qquad \text{a.e. on } \mathbb{R}^d,\end{aligned}$$

where $\mathbf{L}_j(\mathbf{t}) = \big(\mathbf{v}_{(j-1)n+1}(\mathbf{t}), ..., \mathbf{v}_{jn}(\mathbf{t})\big)$ is the $n \times nm$ matrix such that $\mathbf{v}_l$ is the l-th column vector of the matrix $\big(\mathbf{H}_1(\mathbf{t}), ..., \mathbf{H}_m(\mathbf{t})\big)\mathbf{Q}^*(\mathbf{t})$. Observe that for every $j \in \{1, ..., m\}$ the entries of the matrix $\mathbf{L}_j$ are $\mathbb{Z}^d$-periodic functions. Applying the Minkowski and Hölder inequalities, we conclude that $\mathbf{L}_j \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, and the conclusion follows by another application of Lemma 2. $\square$

A straightforward consequence of the preceding propositions is the following.

Corollary 1. *Let* $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}, \mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *and* $\mathbf{T}(\mathbf{G})$ *are orthonormal sequences in a closed subspace* $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. *If* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis for* S, *then* $\mathbf{T}(\mathbf{G})$ *is an orthonormal basis for* S.

Proof. By Proposition 1, there exists a matrix $\mathbf{Q}(\mathbf{t}) := \big(h_{q,r}(\mathbf{t})\big)_{q,r=1}^{nm}$ where $h_{q,r} \in L^2(\mathbb{T}^d)$, which satisfies $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. $(\mathbf{t})$ on $\mathbb{R}^d$ and also (12) holds. Thus, the proof is finished by Proposition 3. $\square$

4. Wavelets and Multiresolution analysis

In what follows we will assume that A is an expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$. Here and further we use the same notation for a linear map on $\mathbb{R}^d$ and its matrix with respect to the canonical base.

In this section we introduce the notions of matrix–valued wavelet set and matrix–valued multiresolution analysis (A-MMRA) associated with a dilation given by a fixed map A as above in a signal space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $d, n \geq 1$. These definitions generalize the matrix–valued wavelet and matrix–valued multiresolution analysis notions defined in [22] when $d = 1$ and A is the dyadic dilation. We study the structure of an A-MMRA, present a strategy to construct matrix–valued wavelet sets associated with a fixed dilation A and characterize the matrix–valued wavelet sets constructed from a given A-MMRA.

Given an expansive linear map A, a matrix-valued wavelet set associated with A is a finite set of functions $\{\Psi_1, ..., \Psi_s\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that the system

$$\{\mathbf{D}_A^j \mathbf{T}_\mathbf{k} \Psi_r : r = 1, 2..., s,\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^d\},$$

is an orthonomal basis for $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$.

A general method for constructing matrix–valued wavelet sets on $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is related to the concept of matrix–valued multiresolution analysis associated with A (A-MMRA): Given an expansive linear map A as above, we define an A-MMRA as a sequence of closed subspaces $V_j,\ j \in \mathbb{Z}$, of $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ that satisfies the following conditions:

(i) For every $j \in \mathbb{Z},\ V_j \subset V_{j+1}$;
(ii) For every $j \in \mathbb{Z},\ \mathbf{f}(\mathbf{x}) \in V_j$ if and only if $\mathbf{f}(A\mathbf{x}) \in V_{j+1}$;
(iii) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$;
(iv) There exists a function $\Phi \in V_0$, called a *scaling function*, such that

$$\{\ \mathbf{T}_\mathbf{k}\Phi(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^n\ \}$$

is an orthonormal basis for V_0.

To construct a matrix–valued wavelet set associated with a dilation map A from an A-MMRA with scaling function Φ, we denote by W_j the orthogonal complement of V_j in V_{j+1}. Thus, by condition (i), we have $V_{j+1} = W_j \oplus V_j$. Moreover, condition (iii) implies that $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n}) = \oplus_{j\in\mathbb{Z}} W_j$.

Observe that by condition (ii) we have

$$\forall j \in \mathbb{Z}, \qquad \mathbf{f}(\cdot) \in W_0 \Leftrightarrow \mathbf{f}(A^j\cdot) \in W_j. \tag{14}$$

Thus, to find a matrix–valued wavelet set from an A-MMRA, it will suffice to construct a set of functions $\{\Psi_1, ..., \Psi_s\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that the system

$$\{\mathbf{T}_\mathbf{k}\Psi_r : r = 1, 2..., s,\ \mathbf{k} \in \mathbb{Z}^d\},$$

is an orthonomal basis for W_0, for then

$$\{\mathbf{D}_A^j \mathbf{T}_\mathbf{k}\Psi_r : r = 1, 2..., s,\ \mathbf{k} \in \mathbb{Z}^d\},$$

is an orthonormal basis of W_j.

We now focus on how to construct orthonormal bases of integer translates for the subspaces V_1 and W_0. For this purpose we study the structure of the subspaces V_j and W_j.

Let us recall that if $A : \mathbb{R}^d \to \mathbb{R}^d$ is an expansive linear map such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, then the quotient group $\mathbb{Z}^d/A(\mathbb{Z}^d)$ is well defined. We will denote by $\Delta_A \subset \mathbb{Z}^d$ a full collection of representatives of the cosets of $\mathbb{Z}^d/A(\mathbb{Z}^d)$. There are exactly d_A cosets (see [10] and [24, p. 109]). Let $\Delta_A = \{\mathbf{q}_i\}_{i=0}^{d_A-1}$ where $\mathbf{q}_0 = \mathbf{0}$.

Note that, if $l \in \{0, 1, 2, ...\}$, then $l = ad_A + i$, where $a \in \{0, 1, 2, ...\}$ and $i \in \{0, 1, ...d_A - 1\}$.

We have:

Theorem 1. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear map such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$. Let $\mathbf{F} = \{\mathbf{f}_0, ..., \mathbf{f}_{m-1}\}$ be a set of functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{F})$ is an orthonormal basis of a closed subspace V of $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, and let $U = \{\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n}) : \mathbf{f}(A^{-1}\cdot) \in V\}$. If*

$$\mathbf{g}_l := d_A^{1/2}\mathbf{f}_a(A\mathbf{x} + \mathbf{q}_i), \qquad l \in \{0, ..., md_A - 1\}$$

then $\mathbf{T}(\mathbf{g}_0, ..., \mathbf{g}_{md_A-1})$ *is an orthonormal basis of* U*. Moreover, any set of functions* $\mathbf{G}$ *in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{G})$ *is an orthonormal basis of* U *has exactly* md_A *functions.*

Proof. Since $\mathbf{T}(\mathbf{F})$ is an orthonormal sequence, a trivial change of variables shows that $\mathbf{T}(\mathbf{g}_0, ..., \mathbf{g}_{md_A-1})$ is an orthogonal sequence. Further, since $\Delta_A = \{\mathbf{q}_i\}_{i=0}^{d_A-1}$ is a full collection of representatives of the cosets of $\mathbb{Z}^d/A(\mathbb{Z}^d)$, given $a \in \{0, ..., m-1\}$ and $\mathbf{k} \in \mathbb{Z}^d$ we have that there exist unique $l \in \{0, ..., m-1\}$ and $\mathbf{r} \in \mathbb{Z}^d$ such that $\mathbf{D}_A\mathbf{T}_\mathbf{k}\mathbf{f}_a = \mathbf{T}_\mathbf{r}\mathbf{g}_l$. Thus $\mathbf{T}(\mathbf{g}_0, ..., \mathbf{g}_{md_A-1})$ is an orthonormal basis of U.

Since the set $\{\mathbf{g}_0, ..., \mathbf{g}_{md_A-1}\}$ has exactly md_A functions, Proposition 2 implies that every other set of functions $\mathbf{G}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal basis of U has exactly md_A functions. $\square$

Theorem 1 yields

Theorem 2. *Let* $\Phi \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *be a scaling function in an A-MMRA,* $\{V_j : j \in \mathbb{Z}\}$*. If*

$$\Theta_i := d_A^{1/2}\Phi(A\mathbf{x} + \mathbf{q}_i), \qquad i = 0, 1, ..., d_A - 1, \tag{15}$$

then $\mathbf{T}(\Theta_0, ..., \Theta_{d_A-1})$ *is an orthonormal basis of* V_1*.*

Using Theorem 1 we can deduce some properties of the subspaces V_j. We have the following.

Theorem 3. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be an A-MMRA. Then*

(a) *If* $j > 0$*, then there exists a finite set* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* V_j*.*
(b) *If* $j \geq 0$*, then any set* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* V_j *has exactly* d_A^j *functions.*
(c) *If* $j < 0$*, then there is no set of functions* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* V_j*.*
(d) *If* $j \neq 0$*, then there is no function* $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{f})$ *is an orthonormal basis of* V_j*.*

Proof. To prove (a), let Φ be a scaling function in the A-MMRA. According to Theorem 2, there exists a set of exactly d_A functions, $\mathbf{F}_1$, such that $\mathbf{T}(\mathbf{F}_1)$ is an orthonormal basis of V_1. Thus for $j \geq 0$ the existence of a set of exactly d_A^j functions, $\mathbf{F}_j$, such that $\mathbf{T}(\mathbf{F}_j)$ is an orthonormal basis of V_j follows by repeated application of Theorem 1.

From (a), for $j \geq 0$ the set $\mathbf{F}_j$ has exactly d_A^j functions; thus (b) follows from Proposition 2.

We now prove (c). Let $m := d_A^{-j}$. By repeated application of Theorem 1 we conclude that there are functions $f_0, \ldots f_{m-1}$ such that $T(f_0, \ldots f_{m-1})$ is a basis of V_0. Since A is expansive, we know that $d_A > 1$; thus $m > 1$, which is a contradiction of (b).

Finally, if $j < 0$ (d) follows from (c), whereas if $j > 0$, (d) follows from (b). $\square$

The following two corollaries are immediate consequences of Theorem 3.

Corollary 2. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be an A-MMRA, let* $s > 0$ *and* $U_j := V_{j+s}$*. Then* $\{U_j : j \in \mathbb{Z}\}$ *is a sequence of closed subspaces in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *satisfying the conditions (i), (ii), (iii) in the definition of A-MMRA, and also, there exists a set* *of functions* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* U_0 *and it has exactly* d_A^s *functions.*

Corollary 3. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be an A-MMRA. Then* V_j *is a proper subset of* V_{j+1} *for every* $j \in \mathbb{Z}$*.*

Proof. Assume that there is $j \in \mathbb{Z}$ such that $V_j = V_{j+1}$, then by the definition of A-MMRA we have $V_j = V_{j+s}$ for every $s \in \mathbb{Z}$. Thus, in particular $V_0 = V_1$ and which is impossible by the condition (b) in Theorem 3. $\square$

We have the following characterization of matrix–valued wavelet sets constructed from an A-MMRA:

Theorem 4. *Let* $\Phi \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *be a scaling function in an A-MMRA,* $\{V_j : j \in \mathbb{Z}\}$*, and let* $\Theta_0, ..., \Theta_{d_A-1} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *be such that* $\mathbf{T}(\Theta_0, ...\Theta_{d_A-1})$ *is an orthonormal basis of* V_1*. The following propositions are equivalent:*

(a) $\{\Psi_1, ..., \Psi_{d_A-1}\}$ *is a matrix–valued wavelet set constructed from the given A-MMRA.*

(b) *There is an* $n\, d_A \times n\, d_A$ *matrix* $\mathbf{Q}(\mathbf{t})$ *of* $\mathbb{Z}^d$*-periodic functions and unitary a.e. on* $\mathbb{R}^d$ *such that*

$$\big(\widehat{\Phi}(\mathbf{t}), \widehat{\Psi}_1(\mathbf{t}), ..., \widehat{\Psi}_{d_A-1}(\mathbf{t})\big)^T := \mathbf{Q}(\mathbf{t})\big(\widehat{\Theta}_0(\mathbf{t}), \widehat{\Theta}_1(\mathbf{t}), ..., \widehat{\Theta}_{d_A-1}(\mathbf{t})\big)^T \quad \textit{a.e. on } \mathbb{R}^d.$$

Proof. Let us prove (a) $\Rightarrow$ (b). The condition (a) means that $\mathbf{T}(\Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of W_0 where W_0 is defined as the orthogonal complement of V_0 in V_1. Further, since $\mathbf{T}(\Phi)$ is an orthonormal basis of V_0 then $\mathbf{T}(\Phi, \Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of V_1. Thus the conditions (b) follows from Proposition 1.

We now prove (b) $\Rightarrow$ (a). According to Proposition 1, we know that

$$\mathbf{T}(\Phi, \Psi_1, ..., \Psi_{d_A-1})$$

is an orthonormal sequence in V_1, and further, by Proposition 3, we know that $\mathbf{T}(\Phi, \Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of V_1. Thus, since $\mathbf{T}(\Phi)$ is an orthonormal basis of V_0 and $V_1 = W_0 \oplus V_0$ then $\mathbf{T}(\Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of W_0. Thus, we conclude that $\{\Psi_1, ..., \Psi_{d_A-1}\}$ is a matrix–valued wavelet set constructed from the A-MMRA. $\square$

We now proceed to describe a strategy for constructing a matrix–valued wavelet set associated to a dilation A from a given A-MMRA with a scaling function Φ. According to Theorem 2 the functions $\Theta_0, \ldots, \Theta_{d_A-1}$ defined by (15) are such that $\mathbf{T}(\Theta_0, \ldots, \Theta_{d_A-1})$ is an orthonormal basis of V_1. Furthermore, since $\Phi \in V_0 \subset V_1$, Lemma 2 implies that there are $\mathbb{Z}^d$-periodic matrix–valued functions $\mathbf{H}_l \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $l = 0, ..., d_A - 1$, such that

$$\widehat{\Phi}(\mathbf{t}) = \sum_{l=0}^{d_A-1} \mathbf{H}_l(\mathbf{t})\widehat{\Theta_l}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d.$$

Moreover, Lemma 3 implies that

$$\sum_{l=0}^{d_A-1} \mathbf{H}_l(\mathbf{t})\mathbf{H}_l^*(\mathbf{t}) = I_n \qquad \text{a.e. on } \mathbb{R}^d, \tag{16}$$

If we denote by $\mathbf{J}_0$ the $n \times nd_A$ matrix of functions defined by

$$\mathbf{J}_0(\mathbf{t}) = (\mathbf{H}_0(\mathbf{t}), ..., \mathbf{H}_{d_A-1}(\mathbf{t}))$$

and by $\mathbf{v}_q(\mathbf{t})$, $q = 1, ..., n$ the vector in the complex vector space $\mathbb{C}^{nd_A}$ defined by the value at $\mathbf{t}$ of the q-th row in the matrix $\mathbf{J}_0(\mathbf{t})$, the equality (16) implies that the vectors $\{\mathbf{v}_q(\mathbf{t}) : q = 1, ..., n\}$ are a.e. orthonormal. Note that it is possible to construct a $nd_A \times nd_A$ matrix $\mathbf{Q}(\mathbf{t})$ of $\mathbb{Z}^d$-periodic functions, a.e. unitary in $\mathbb{R}^d$, such that for $q = 1, \ldots, n$ the q-th row is given by the function vector $\mathbf{v}_q(\mathbf{t})$. The construction of such a matrix can be done by the Gram–Schmidt orthogonalization process. If $\mathbf{H}_1(\mathbf{t})$ is symmetric, another method for the completion of a unitary matrix is given in [17]. Finally, if $\Psi_1, ..., \Psi_{d_A-1} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is defined by

$$\left(\widehat{\Phi}(\mathbf{t}), \widehat{\Psi}_1(\mathbf{t}), \cdots, \widehat{\Psi}_{d_A-1}(\mathbf{t})\right)^T = \mathbf{Q}(\mathbf{t})\left(\widehat{\Theta}_0(\mathbf{t}), \widehat{\Theta}_1(\mathbf{t}), \ldots, \widehat{\Theta}_{d_A-1}(\mathbf{t})\right)^T \quad \text{a.e. on } \mathbb{R}^d,$$

and applying Theorem 4, we conclude that $\{\Psi_1, ..., \Psi_{d_A-1}\}$ is a matrix–valued wavelet set constructed from the given A-MMRA.

We have therefore proved the following.

Theorem 5. *Given an expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and given an A-MMRA, then there exists a set of matrix–valued functions $\{\Psi_1, ..., \Psi_{d_A-1}\}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ which is a matrix–valued wavelet set constructed from such an A-MMRA.*

Recalling that a set of matrix–valued functions $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is a matrix–valued wavelet set constructed from an A-MMRA, $\{V_j : j \in \mathbb{Z}\}$, if and only if $\mathbf{T}(\mathbf{F})$ is an orthonormal basis of the subspace W_0 defined as the orthogonal complement of V_0 in V_1, then the following is a corollary of Theorem 5 and Proposition 2.

Corollary 4. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let W_0 denote the orthogonal complement of V_0 in V_1. Then there exists a set of matrix–valued functions $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{F})$ is an orthonormal basis of W_0, and any set of matrix–valued functions $\mathbf{G}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal basis of W_0 has exactly $d_A - 1$ matrix–valued functions.*

From Corollary 4, (14), and Theorem 1, we obtain the following.

Corollary 5. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let W_j denote the orthogonal complement of V_j in V_{j+1}. Then, for every $j \in \{0, 1, 2, ...\}$ there exists a set of matrix–valued functions $\mathbf{F}_j \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{F}_j)$ is an orthonormal basis of W_j, and any set of functions $\mathbf{G}_j$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G}_j)$ is an orthonormal basis of W_j has exactly $(d_A - 1)d_A^j$ matrix–valued functions.*

Let us continue with the study of the structure of subspaces V_j and W_j.

Theorem 6. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_{d_A-1}\}$ be a set of matrix–valued functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If there exists an integer $l < 0$ such that $\mathbf{F} \subset V_l$, then $\mathbf{F}$ cannot be a matrix–valued wavelet set.*

Proof. If $\mathbf{F}$ is a matrix–valued wavelet set then $\mathbf{T}(\mathbf{F})$ is an orthonormal sequence in V_l. Thus, applying Theorem 1 with the expansive linear map A^l, we see that there exist a set of $(d_A - 1)d_A^l$ matrix–valued functions $\mathbf{G}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal sequence in V_0. Moreover, according to the definition of V_0 and Proposition 2, the number $(d_A - 1)d_A^l$ must be less or equal to 1. Since $d_A \geq 2$, we have a contradiction. □

Theorem 7. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_{d_A-1}\}$ be a set of matrix–valued functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If there exists an integer $l \neq 0$ such that $\mathbf{F} \subset W_l$, then $\mathbf{F}$ cannot be a matrix–valued wavelet set.*

Proof. Assume that $\mathbf{F}$ is a matrix–valued wavelet set. If $l < 0$, since W_l is a proper subset of V_0 it follows that $d_A - 1$ must be less of equal to 1 and this is impossible. On the other hand, if $l > 0$, Corollary 5 implies that every set of matrix–valued functions $\mathbf{G} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal basis of W_l must have exactly $(d_A - 1)d_A^l$ matrix–valued functions. Since $d_A < (d_A - 1)d_A^l$ we get a contradiction in this case as well. □

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